Swiss Federal Institute of Technology (ETHZ)

Physics Department

Master's Course:

"Scattering Amplitudes in Quantum Field Theories"

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Foreword

Scattering amplitudes are central to particle physics: their modulus square is the most important ingredient in the computation of cross sections. In the last few years, they have been assuming an increasingly relevant role also in gravitational scattering, e.g. in black-body scattering with emission of gravitational radiation.

In particle physics, the traditional approach to the computation of an unpolarised cross section is to square the amplitude and sum over the polarisations of the external states. The outcome is an expression in terms of Mandelstam invariants and masses. The traditional workflow:
Lagrangian → Feynman rules → Feynman diagrams → scattering amplitude → squared amplitude → cross section
has a bottleneck in squaring the amplitude, because if n Feynman diagrams contribute to the amplitude, n^2 terms will appear in the square of the amplitude. The computation becomes quickly intractable as the number of external particles grows.

With gluons only, here is the number of Feynman diagrams in the amplitude as a function of the number of external gluons:

# external gluons	# Feynman diagrams
4	4
6	220
8	34,300
10	10,521,900

Then we must square the amplitude: the number of terms becomes quickly intractable even with state-of-the-art computers.

Fixing the polarisations ,i.e. for massless particles the helicities, of the external states, improves a lot the computation: for a given helicity configuration, the amplitude is a complex number. Then we just square that number.

Further, different helicity configurations do not interfere in the squared amplitude, summed over the polarisations. So if an amplitude has m helicity configurations, the squared amplitude, summed over polarisations, is the sum of m squared amplitudes at fixed helicities.

Last but not least, the amplitude will turn out to be much simpler than what might have been guessed for the sum of n Feynman diagrams. For some helicity configurations in particular, the maximally helicity violating (MHV) ones, amplitudes are just a monomial function of kinematic invariants.

Amplitudes at fixed helicities were introduced in the 80's. For massless fermions, they are convenient because helicity and chirality coincide, and chirality is preserved on massless-fermion lines. That reduces the number of helicity configurations to be computed. Further, it was found that fixed helicities were convenient also to represent external photons or gluons. Finally, what really changed the game was the discovery in 1986 by Parke and Taylor [1] that MHV amplitudes for the scattering of an arbitrary number of gluons are written as just one term, a rational function of kinematic invariants. Shortly after, it was realised that amplitudes are organised in complexity according to the degree of helicity violation: the simplest are the MHV amplitudes($\pm\pm \ldots \pm\mp\mp$), the next to simplest are the next-to-MHV (NMHV) ($\pm\pm \ldots \pm\mp\mp\mp$), and so on.

Ultimately, one might want to re-think the role of quantum field theories in particle physics emphasising on-shell structures, keeping in mind that the fundamental pillars which any possible on-shell formulation of particle physics will share with quantum field theories are quantum mechanics and special relativity.

In the first part of these lectures, we will review these developments and the subsequent event in this story, the Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion relations [2].

Helicity amplitudes have started making their way in QFT textbooks. They can be found in [3, 4, 5, 6].

There are also dedicated reviews on the topic. The first, and still very informative, is [7] and then the lectures by [8, 9, 10], the SAGEX reviews, in particular the first [11], and a review on colour-kinematics duality [12].

Finally, there are a few books on scattering amplitudes [13, 14, 15].

I will pick up threads, ideas and examples from all of the sources above.

Chapter 1

Tree amplitudes

1.1 One-particle states

Amplitudes scatter particles, which are on-shell states. Let us see then how we can characterise quantum one-particle states. They can be classified according to how they transform under the inhomogeneous Lorentz (or Poincaré) group. In this presentation, we follow sec. 2.5 of ref. [16].

The components of the momentum commute with each other, so they can be chosen to express a particle as an eigenvector of momentum. The discrete degrees of freedom (like helicity, or possibly other quantum numbers) will be labeled by σ. So the one-particle state can be written as $|p;\sigma\rangle$ with

$$P^\mu |p;\sigma\rangle = p^\mu |p;\sigma\rangle \ . \tag{1.1}$$

We assume that for every transformation Λ of the Lorentz group, we have a unitary operator $U(\Lambda)$ acting on the Hilbert space, such that $U(\Lambda_1\Lambda_2) = U(\Lambda_1)U(\Lambda_2)$.

The Lorentz transformation properties of P^μ are given by

$$U(\Lambda)P^\mu U^{-1}(\Lambda) = \Lambda_\nu{}^\mu P^\nu \,, \tag{1.2}$$

which says that under Lorentz transformations P^μ transforms as a vector (remember that $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ and $(\det\Lambda)^2 = 1$ imply that $\Lambda^\mu{}_\nu$ has an inverse $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu = \eta_{\nu\rho}\eta^{\mu\sigma}\Lambda^\rho{}_\sigma$).

The effect of acting on $|p;\sigma\rangle$ with the operator $U(\Lambda)$ is to yield an eigenvector of P^μ with eigenvalue Λp,

$$\begin{aligned} P^\mu U(\Lambda)|p;\sigma\rangle &= U(\Lambda)(U^{-1}(\Lambda)P^\mu U(\Lambda))|p;\sigma\rangle \\ &= U(\Lambda)(\Lambda^{-1})_\nu{}^\mu P^\nu |p;\sigma\rangle \\ &= \Lambda^\mu{}_\nu p^\nu U(\Lambda)|p;\sigma\rangle \ . \end{aligned} \tag{1.3}$$

So $U(\Lambda)\left|p;\sigma\right\rangle$ must be a linear combination of Lorentz-transformed one-particle states,

$$U(\Lambda)\left|p;\sigma\right\rangle = C_{\sigma\sigma'}(\Lambda,p)\left|\Lambda p;\sigma'\right\rangle\,, \tag{1.4}$$

where it is understood that the index σ' is summed over. The issue is then to express the coefficients $C_{\sigma\sigma'}(\Lambda,p)$ in terms of the irreducible representations of the Poincaré group.

The only functions of p^μ which are left invariant by (proper orthochronous) Lorentz transformations are $p^2=m^2$ and, if $m^2>0$, the sign of p^0. For each p^2, we can choose a reference momentum k^μ, such that $p^\mu = L^\mu{}_\nu(p;k)k^\nu$ and we can define the one-particle states as

$$\left|p;\sigma\right\rangle = U(L(p;k))\left|k;\sigma\right\rangle\,. \tag{1.5}$$

We can see how $\left|p;\sigma\right\rangle$ transforms under general Lorentz transformations,

$$\begin{aligned} U(\Lambda)\left|p;\sigma\right\rangle &= U(\Lambda)\;U(L(p;k))\left|k;\sigma\right\rangle \\ &= U(L(\Lambda p;k))\;U\underbrace{\left(L^{-1}(\Lambda p;k)\Lambda L(p;k)\right)}_{W}\left|k;\sigma\right\rangle\,. \end{aligned} \tag{1.6}$$

Now, $W = L^{-1}(\Lambda p;k)\Lambda L(p;k)$ maps k to $p = L(p;k)\;k$, then to $\Lambda\;p$, then back to k, so it leaves k^μ invariant,

$$W^\mu{}_\nu k^\nu = k^\mu\,. \tag{1.7}$$

The subgroup of the Lorentz group made of the Lorentz transformations which leave k^μ invariant is the little group.

The action of W on the reference state $\left|k;\sigma\right\rangle$ yields a linear combination of reference states,

$$U(W(\Lambda,p;k))\left|k;\sigma\right\rangle = D_{\sigma\sigma'}(W(\Lambda,p;k))\left|k;\sigma'\right\rangle\,, \tag{1.8}$$

where the coefficients $D_{\sigma\sigma'}$ provide a representation of the little group. But,

$$\begin{aligned} U(\Lambda)\left|p;\sigma\right\rangle &= U(L(\Lambda p;k))U(W(\Lambda,p;k))\left|k;\sigma\right\rangle \\ &= D_{\sigma\sigma'}(W(\Lambda,p;k))U(L(\Lambda p;k))\left|k;\sigma'\right\rangle \\ &= D_{\sigma\sigma'}(W(\Lambda,p;k))\left|\Lambda p;\sigma'\right\rangle\,. \end{aligned} \tag{1.9}$$

Comparing it to eq. (1.4), we see that the issue of determining the coefficients $C_{\sigma\sigma'}(\Lambda,p)$ has been reduced to the issue of finding the representations of the little group.

Therefore, a particle transforms under representations of the little group. An n-point scattering amplitude M_n is then labelled by the one-particle states, $\left|p_i;\sigma_i\right\rangle$ with $i=1,\ldots,n$. Poincaré invariance implies that

$$\mathcal{M}_n(p_i;\sigma_i) = \delta^D(p_1^\mu + \cdots + p_n^\mu) M_n(p_i;\sigma_i)\,, \tag{1.10}$$

$$M_n^\Lambda(p_i;\sigma_i) = \prod_{i=1}^{n} D_{\sigma_i\sigma_i'}(W) M_n((\Lambda p)_i;\sigma_i')\,. \tag{1.11}$$

For massive particles, as reference momentum we choose $k^\mu = (m,0,0,0)$, and it is fairly obvious that the only Lorentz transformations which leave invariant a massive particle at rest are the rotations. So in four dimensions, the little group is $SO(3)$, which is isomorphic to $SU(2)$.

For massless particles, as reference momentum we may choose a particle in any direction, say in the beam axis, so that $k^\mu = (E,0,0,E)$. Then it is shown in app. H.2 that the little group is $ISO(2)$, i.e. the group of two-dimensional rotations about the beam axis, $SO(2)$, which is isomorphic to $U(1)$, complemented by two translations.

In D dimensions, these groups become $SO(D-1)$ in the massive case, and $SO(D-2)$ complemented by $(D-2)$ translations in the massless case.

In the massless case in four dimensions, the generator of rotations about the beam axis is J_z and its eigenvalue is the helicity,

$$J_z\,|k;\sigma\rangle = \sigma\,|k;\sigma\rangle\ , \tag{1.12}$$

while the generators of translations are ditched, because they generate continuous eigenvalues which are not observed in Nature, so the eigenvectors are taken with null eigenvalues: see sec. 2.5 of ref. [16]. Thus, massless particles in four dimensions are labelled by their momentum and helicity.

1.2 Spinor-helicity formalism

Let us introduce the conventions that we use:

- Light-cone coordinates: $p^\pm = p^0 \pm p^3$
- Complex tranverse momentum: $p_\perp = p^1 + ip^2$

such that $2p\cdot q = p^+q^- + p^-q^+ - p_\perp q_\perp^* - p_\perp^* q_\perp$ and $p^2 = p^+p^- - p_\perp p_\perp^*$.

For a light-like momentum: $p^2 = 0 \Rightarrow p^+p^- = p_\perp p_\perp^*$.

The Pauli matrices are used in the combination:

$$\sigma^+ = \tfrac{1}{2}(1+\sigma^3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \tfrac{1}{2}(1-\sigma^3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\sigma_\perp = \tfrac{1}{2}(\sigma^1 + i\sigma^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \overline{\sigma}_\perp = \tfrac{1}{2}(\sigma^1 - i\sigma^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

such that if $\sigma^\mu = (1,\vec{\sigma})$ and $\overline{\sigma}^\mu = (1,-\vec{\sigma})$, then

$$p\cdot\sigma = p^0\mathbb{1} - \vec{p}\cdot\vec{\sigma} = p^+\sigma^- + p^-\sigma^+ - p_\perp\overline{\sigma}_\perp - p_\perp^*\sigma_\perp = \begin{pmatrix} p^- & -p_\perp^* \\ -p_\perp & p^+ \end{pmatrix},$$

$$p\cdot\overline{\sigma} = p^0\mathbb{1} + \vec{p}\cdot\vec{\sigma} = p^+\sigma^+ + p^-\sigma^- + p_\perp\overline{\sigma}_\perp + p_\perp^*\sigma_\perp = \begin{pmatrix} p^+ & p_\perp^* \\ p_\perp & p^- \end{pmatrix}.$$

Note that $p^2 = \det(p\cdot\sigma) = \det(p\cdot\overline{\sigma})$ or that $p^2\mathbb{1} = (p\cdot\sigma)(p\cdot\overline{\sigma})$. So for $p^2 \neq 0$, $p\cdot\sigma$ and $p\cdot\overline{\sigma}$ have rank 2; for $p^2 = 0$, $p\cdot\sigma$ and $p\cdot\overline{\sigma}$ have rank < 2.

1.2.1 Massless spinors

Let us introduce **Dirac** spinors $u = \begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix}$ with 2-dimensional **Weyl** spinors $\xi_\pm$, the **helicity** operator,

$$h = \frac{\vec{p}\cdot\vec{\Sigma}}{2|\vec{p}|} \qquad \text{with} \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{1.13}$$

and spinors of definite helicity,

$$u_\mp = \frac{1\mp\gamma^5}{2}u, \qquad \overline{u}_\pm = \overline{u}\frac{1\mp\gamma^5}{2}. \tag{1.14}$$

We use the chiral representation of the gamma matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.15}$$

such that

$$u_+ = \frac{1+\gamma^5}{2}u = \begin{pmatrix} 0 \\ \xi_+ \end{pmatrix}, \qquad u_- = \frac{1-\gamma^5}{2}u = \begin{pmatrix} \xi_- \\ 0 \end{pmatrix}. \tag{1.16}$$

The massless Dirac equation is

$$p_\mu\gamma^\mu u(p) = 0\,, \tag{1.17}$$

which, in chiral representation, becomes

$$p_\mu \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix} = \begin{pmatrix} p\cdot\sigma\,\xi_+ \\ p\cdot\overline{\sigma}\,\xi_- \end{pmatrix} = 0\,. \tag{1.18}$$

For massless spinors, the normalisation is $\xi_-^\dagger\xi_- = \xi_+^\dagger\xi_+ = 2E$, where $E = p^0 = |\vec{p}|$. From (1.18), we then have to solve two systems of equations. The first,

$$p\cdot\sigma\,\xi_+ = \begin{pmatrix} p^- & -p_\perp^* \\ -p_\perp & p^+ \end{pmatrix}\xi_+ = 0\,, \tag{1.19}$$

yields two equations for the components of ξ_+ which are linearly dependent. The solution is

$$\xi_+ = e^{i\alpha}\begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-}e^{i\phi_p} \end{pmatrix}, \qquad p^+ \neq 0\,, \tag{1.20}$$

with

$$e^{i\phi_p} = \frac{p_\perp}{\sqrt{p^+p^-}} = \frac{p_\perp}{\sqrt{p_\perp p_\perp^*}} = \sqrt{\frac{p_\perp}{p_\perp^*}}\,, \tag{1.21}$$

where ξ_+ is normalised in such a way that $\xi_+^\dagger \xi_+ = 2E$ and $e^{i\alpha}$ is an overall arbitrary phase. The second system from (1.18) is

$$p \cdot \overline{\sigma} \xi_- = \begin{pmatrix} p^+ & p_\perp^* \\ p_\perp & p^- \end{pmatrix} \xi_- = 0\,, \tag{1.22}$$

with solution

$$\xi_- = e^{i\beta} \begin{pmatrix} -\sqrt{p^-} e^{-i\phi_p} \\ \sqrt{p^+} \end{pmatrix}, \qquad p^+ \neq 0\,, \tag{1.23}$$

with $e^{i\beta}$ an overall arbitrary phase.

Neglecting the arbitrary phases, if $p^+ = 0$ the on-shell condition $p^+p^- = p_\perp p_\perp^*$ implies that $p_\perp = 0$, so ξ_+ and ξ_- become

$$\xi_+ = \begin{pmatrix} 0 \\ \sqrt{p^-} \end{pmatrix}, \qquad \xi_- = \begin{pmatrix} -\sqrt{p^-} \\ 0 \end{pmatrix}, \qquad \text{for} \qquad p^+ = 0\,. \tag{1.24}$$

For $p^+ \neq 0$, we have that

$$\xi_+ = \frac{1}{\sqrt{p^+}} \begin{pmatrix} p^+ \\ p_\perp \end{pmatrix}, \qquad \xi_- = \frac{1}{\sqrt{p^+}} \begin{pmatrix} -p_\perp^* \\ p^+ \end{pmatrix}. \tag{1.25}$$

The phase convention has been chosen such that

$$\xi_+ = i\sigma^2 \xi_-^*\,, \tag{1.26}$$

or inverting

$$\xi_-^* = -i\sigma^2 \xi_+\,. \tag{1.27}$$

$-i\sigma^2$ is the 2-dimensional realisation of the 4-dimensional charge conjugation matrix $C = -i\gamma^2$ (see app. H.1) that maps a fermion of a given spin into the anti-fermion of the same spin (see e.g. [3], ch. 3). It is defined by $C\gamma_\mu^* C^{-1} = -\gamma_\mu$, which in 2 dimensions becomes $\sigma_2 \overline{\sigma}_\mu \sigma_2 = \sigma_\mu^T$, a property of the Pauli matrices we will use over and over.

Because positive $u(k)\overline{u}(k)$ and negative $v(k)\overline{v}(k)$ energy projection operators are both proportional to $\not{k}$ for $k^2 = 0$, solutions of definite helicity,

$$u_\pm(k) = \frac{1 \pm \gamma_5}{2} u(k)\,, \qquad v_\mp(k) = \frac{1 \pm \gamma_5}{2} v(k)\,, \tag{1.28}$$

can be chosen to be equal, $u_\pm(k) = v_\mp(k)$. Note that for massless particles of positive energy, helicity and chirality coincide. For negative energy, the helicity is the opposite of the chirality.

1.2.2 Spinor products

For $p^+, k^+ \neq 0$ and $p^0, k^0 > 0$, we can construct right-handed spinor products,

$$\langle kp\rangle \equiv \left\langle k^- \middle| p^+ \right\rangle = \overline{u_-}(k)u_+(p) = \xi_-^\dagger(k)\xi_+(p) = -k_\perp\sqrt{\frac{p^+}{k^+}} + p_\perp\sqrt{\frac{k^+}{p^+}}\,, \tag{1.29}$$

and left-handed spinor products,

$$[kp] \equiv \left\langle k^+ \middle| p^- \right\rangle = \overline{u_+}(k)u_-(p) = \xi_+^\dagger(k)\xi_-(p) = k_\perp^*\sqrt{\frac{p^+}{k^+}} - p_\perp^*\sqrt{\frac{k^+}{p^+}}\,. \tag{1.30}$$

The spinor products have the properties of:

1. **Antisymmetry**:

$$\langle pp\rangle = [pp] = 0\,, \qquad \langle kp\rangle = -\langle pk\rangle\,, \qquad [kp] = -[pk]\,. \tag{1.31}$$

   In fact, using that $\xi_-^* = -i\sigma^2\xi_+$, we have

$$\langle kp\rangle = \xi_-^\dagger(k)\xi_+(p) = (-i\sigma^2\xi_+(k))^T\xi_+(p) = \xi_+^T(k)(i\sigma^2)\xi_+(p)\,, \tag{1.32}$$

   since $(\sigma^2)^T = -\sigma^2$. Also $< kp >$ is a scalar: its transpose is the same quantity. Therefore

$$\langle kp\rangle = -\xi_+^T(p)(i\sigma^2)\xi_+(k) = (i\sigma^2\xi_+(p))^T\xi_+(k) = -\xi_-^\dagger(p)\xi_+(k) = -\langle pk\rangle\,, \tag{1.33}$$

   i.e. the antisymmetry of σ^2 implies the antisymmetry of the spinor product.

2. **Projection operator:**

$$\left|p^\pm\right\rangle\left\langle p^\pm\right| = \frac{1\pm\gamma_5}{2}\not{p}\,, \tag{1.34}$$

   thus

$$\not{p} = \left|p^+\right\rangle\left\langle p^+\right| + \left|p^-\right\rangle\left\langle p^-\right|\,. \tag{1.35}$$

3. **Squaring:**

$$\langle pk\rangle[kp] = \mathrm{tr}(\frac{1-\gamma_5}{2}\not{p}\not{k}) = 2p\cdot k\,, \tag{1.36}$$

   which follows from multiplying two projection operators.

4. **Complex conjugation:**

$$\langle kp\rangle^* = [pk]\,, \tag{1.37}$$

   which follows from the definition,

$$[pk] = \overline{u_+}(p)u_-(k) = \xi_+^\dagger(p)\xi_-(k) = \xi_-^T(k)\xi_+^*(p) = \xi_-^*(k)^\dagger\xi_+^*(p) = \langle kp\rangle^*\,. \tag{1.38}$$

   It entails that $\langle pk\rangle = \sqrt{s_{pk}}e^{i\phi}$, for some phase ϕ.

5. **Gordon identity:**

$$\left\langle p^{\pm}\right| \gamma^{\mu} \left|p^{\pm}\right\rangle = 2p^{\mu} \,, \tag{1.39}$$

which follows from the definition, and from the explicit representation of the 2-dim spinors, e.g.

$$\left\langle p^{+}\right| \gamma^{\mu} \left|p^{+}\right\rangle = \overline{u_{+}}(p)\gamma^{\mu}u_{+}(p) = \xi_{+}^{\dagger}(p)\sigma^{\mu}\xi_{+}(p) \tag{1.40}$$

then using eq. (1.25) we see that $\xi_{+}^{\dagger}(p)\sigma^{\mu}\xi_{+}(p) = 2p^{\mu}$ (see eq. H.3).

6. **Charge conjugation of currents:**

$$\left\langle p^{+}\right| \gamma^{\mu} \left|k^{+}\right\rangle = \left\langle k^{-}\right| \gamma^{\mu} \left|p^{-}\right\rangle \,, \tag{1.41}$$

which follows from using $\xi_{-}^{*} = -i\sigma^{2}\xi_{+}$ and from the identity $\sigma_{2}\overline{\sigma}_{\mu}\sigma_{2} = \sigma_{\mu}^{T}$ (see eq.(H.8)), e.g.

$$\begin{aligned}
&\left\langle k^{-}\right| \gamma^{\mu} \left|p^{-}\right\rangle = u_{-}^{\dagger}(k)\gamma^{\mu}u_{-}(p) = \xi_{-}^{\dagger}(k)\overline{\sigma}^{\mu}\xi_{-}(p) \\
&= (-i\sigma^{2}\xi_{+}(k))^{T}\overline{\sigma}^{\mu}(-i\sigma^{2})^{*}\xi_{+}^{*}(p) \\
&= \xi_{+}^{T}(k)i\sigma^{2}\overline{\sigma}^{\mu}(-i\sigma^{2})\xi_{+}^{*}(p) \\
&= \xi_{+}^{T}(k)(\sigma^{\mu})^{T}\xi_{+}^{*}(p) \\
&= \xi_{+}^{\dagger}(p)\sigma^{\mu}\xi_{+}(k) = \left\langle p^{+}\right| \gamma^{\mu} \left|k^{+}\right\rangle \,.
\end{aligned} \tag{1.42}$$

7. **Complex conjugation of currents:**

$$\left(\left\langle p^{\pm}\right| \gamma^{\mu} \left|k^{\pm}\right\rangle\right)^{*} = \left\langle k^{\pm}\right| \gamma^{\mu} \left|p^{\pm}\right\rangle \,, \tag{1.43}$$

which follows from the definition, e.g.

$$\left(\left\langle p^{-}\right| \gamma^{\mu} \left|k^{-}\right\rangle\right)^{*} = (\xi_{-}^{\dagger}(p)\overline{\sigma}^{\mu}\xi_{-}(k))^{\dagger} = \xi_{-}^{\dagger}(k)\overline{\sigma}^{\mu}\xi_{-}(p) = \left\langle k^{-}\right| \gamma^{\mu} \left|p^{-}\right\rangle \,. \tag{1.44}$$

8. **Fierz rearrangement:**

$$\left\langle k^{-}\right| \gamma^{\mu} \left|p^{-}\right\rangle \left\langle v^{-}\right| \gamma_{\mu} \left|q^{-}\right\rangle = 2\langle kv\rangle[qp] \,, \tag{1.45}$$

which follows from using $\xi_{+} = i\sigma^{2}\xi_{-}^{*}$ and from the Fierz identity for Pauli matrices,

$$\begin{aligned}
(\sigma^{\mu})^{\dot{a}a}(\sigma_{\mu})^{\dot{b}b} &= 2(i\sigma^{2})^{ab}(i\sigma^{2})^{\dot{a}\dot{b}} \,, \\
(\overline{\sigma}^{\mu})_{a\dot{a}}(\overline{\sigma}_{\mu})_{b\dot{b}} &= 2(i\sigma^{2})_{ab}(i\sigma^{2})_{\dot{a}\dot{b}} \,.
\end{aligned} \tag{1.46}$$

Thus (see also eq. H.27)

$$
\begin{aligned}
&\left\langle k^{-}\right|\gamma^{\mu}\left|p^{-}\right\rangle\left\langle v^{-}\right|\gamma_{\mu}\left|q^{-}\right\rangle \\
&=\xi_{-}^{\dagger}(k)\overline{\sigma}^{\mu}\xi_{-}(p)\xi_{-}^{\dagger}(v)\overline{\sigma}_{\mu}\xi_{-}(q) \\
&=\xi_{-}^{*a}(k)(\overline{\sigma}^{\mu})_{a\dot{a}}\xi_{-}^{\dot{a}}(p)\xi_{-}^{*b}(v)(\overline{\sigma}_{\mu})_{b\dot{b}}\xi_{-}^{\dot{b}}(q) \qquad \text{in components} \\
&=2\xi_{-}^{*a}(k)(i\sigma^{2})_{ab}\xi_{-}^{*b}(v)\xi_{-}^{\dot{a}}(p)(i\sigma^{2})_{\dot{a}\dot{b}}\xi_{-}^{\dot{b}}(q) \\
&=2\xi_{-}^{\dagger}(k)\xi_{+}(v)(-i\sigma^{2}\xi_{-}^{\dagger}(p))^{T}\xi_{-}(q) \\
&=-2\xi_{-}^{\dagger}(k)\xi_{+}(v)\xi_{+}^{\dagger}(p)\xi_{-}(q) \\
&=2\langle kv\rangle[qp]\,.
\end{aligned}
\tag{1.47}
$$

Likewise

$$\left\langle p^{+}\right|\gamma^{\mu}\left|k^{+}\right\rangle\left\langle q^{+}\right|\gamma_{\mu}\left|v^{+}\right\rangle=2[pq]\langle vk\rangle\,. \tag{1.48}$$

For $v=q$, we use the Gordon identity to get,

$$
\begin{aligned}
\left\langle k^{-}\right|\not{q}\left|p^{-}\right\rangle&=\langle kq\rangle[qp]\,,\\
\left\langle p^{+}\right|\not{q}\left|k^{+}\right\rangle&=[pq]\langle qk\rangle\,.
\end{aligned}
\tag{1.49}
$$

Note that the current is nilpotent,

$$
\begin{aligned}
&(\left\langle p^{\pm}\right|\gamma^{\mu}\left|p^{\pm}\right\rangle)^{2}=\left|\left\langle p^{\pm}\right|\gamma^{\mu}\left|p^{\pm}\right\rangle\right|^{2}=4p^{2}=0\,,\\
&\left\langle k^{\pm}\right|\gamma^{\mu}\left|p^{\pm}\right\rangle\left\langle k^{\pm}\right|\gamma_{\mu}\left|p^{\pm}\right\rangle=0\,.
\end{aligned}
\tag{1.50}
$$

but

$$\left\langle k^{\pm}\right|\gamma^{\mu}\left|p^{\pm}\right\rangle\left\langle k^{\pm}\right|\gamma_{\mu}\left|p^{\pm}\right\rangle^{*}=2\langle kp\rangle[kp]=-2p\cdot k\,. \tag{1.51}$$

9. **Schouten identity:**

Since spinors are two-dimensional objects,, any spinor $|k^{+}\rangle$ can be written as a combination of two spinors $|q^{+}\rangle$ and $|v^{+}\rangle$,

$$|k^{+}\rangle=\frac{\langle kv\rangle}{\langle qv\rangle}|q^{+}\rangle+\frac{\langle qk\rangle}{\langle qv\rangle}|v^{+}\rangle\,, \tag{1.52}$$

which can be checked by contracting with $\langle k^{-}|$, $\langle q^{-}|$ or $\langle v^{-}|$. By contracting with another spinor $\langle p^{-}|$, we obtain Schouten identity,

$$\langle pk\rangle\langle qv\rangle+\langle pq\rangle\langle vk\rangle+\langle pv\rangle\langle kq\rangle=0\,, \tag{1.53}$$

which is cyclic in k,q,v. Of course, the identity also holds for square brackets,

$$[pk][qv]+[pq][vk]+[pv][kq]=0\,. \tag{1.54}$$

10. **Momentum conservation**

For an n-point amplitude, momentum conservation (with all momenta incoming or outgoing) is $\sum_{i=1}^{n} p_i^\mu = 0$, which implies that

$$\sum_{i=1}^{n} |p_i^\pm\rangle\langle p_i^\pm| = 0\,. \tag{1.55}$$

Contracting with two spinors of momentum q and k, we obtain one more identity,

$$\sum_{i=1}^{n} [q p_i]\langle p_i k\rangle = 0, \qquad p_i \neq q, k\,. \tag{1.56}$$

Since we take all momenta of an n-point amplitude as, say, outgoing, $\sum_{i=1}^{n} p_i^\mu = 0$, in the crossing to the physical region, $p_1^\mu + p_2^\mu = \sum_{i=3}^{n} p_i^\mu$, the incoming partons must be continued to negative energy. So, we can define the spinor product $\langle pq\rangle$ as in the positive energy case, but with p replaced by -p, if $p^0 < 0$, and likewise for q, and with an extra overall factor of i for each negative energy,

$$u_\pm(-p) = i u_\pm(p)\,. \tag{1.57}$$

Thus, spinor products and currents acquire a sign factor,

$$\begin{aligned} \langle kp\rangle^* &= \mathrm{sign}(k^0 p^0)[pk]\,, \\ \left\langle p^+ \right| \gamma^\mu \left| k^+ \right\rangle^* &= \mathrm{sign}(k^0 p^0) \left\langle k^+ \right| \gamma^\mu \left| p^+ \right\rangle\,. \end{aligned} \tag{1.58}$$

Note that a different representation of the γ matrices entails different Dirac spinors, but the spinor products are the same, up to a phase factor overall (see, e.g. [8] where the Dirac's representation of the γ's is used).

1.2.3 Spinor representations

At the beginning of this section. we wrote

$$p\cdot\overline{\sigma} = p^0 \mathbb{1} + \vec{p}\cdot\vec{\sigma} = p^+\sigma^+ + p^-\sigma^- + p_\perp \overline{\sigma}_\perp + p_\perp^* \sigma_\perp = \begin{pmatrix} p^+ & p_\perp^* \\ p_\perp & p^- \end{pmatrix} \tag{1.59}$$

with $\det(p\cdot\overline{\sigma}) = p^2$. Now $(p\cdot\overline{\sigma})^\dagger = p\cdot\overline{\sigma}$, so any real 4-vector is bijective to a 2×2 Hermitian matrix. Hermiticity is preserved under the mapping,

$$p\cdot\overline{\sigma} \to \tau p\cdot\overline{\sigma}\tau^\dagger\,, \tag{1.60}$$

with τ an arbitrary complex 2×2 matrix. Further, $\det(p\cdot\overline{\sigma}) = p^2$ is preserved if $|\det(\tau)| = 1$. Each τ with $\det(\tau) = 1$ defines a real linear mapping of p^μ with $\det(p\cdot\overline{\sigma})$ invariant, i.e. a homogeneous Lorentz mapping,

$$\tau p\cdot\overline{\sigma}\tau^\dagger \to [\Lambda^\mu_\nu(\tau) p^\nu]\overline{\sigma}_\mu\,. \tag{1.61}$$

τ with $\det(\tau) = 1$ defines $SL(2,\mathbb{C})$, however τ and $-\tau$ define the same Lorentz mapping, so the Lorentz group is equivalent to $SL(2,\mathbb{C})/\mathbb{Z}_2$, with 3 complex parameters = 6 real ones (see [16], sec. 2.7).

In fact, $SL(2,\mathbb{C}) = SU(2) + iSU(2)$. Since the finite dimensional representations of $SU(2)$ are equivalent to the finite dimensional representations of $SL(2,\mathbb{R})$, in the literature we find the (finite dimensional) representations of the Lorentz group expressed by the representations of $SU(2)\otimes SU(2)$, or by the ones of $SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$. Then the finite-dimensional representations are labelled by a pair of indices (i,j) taking integer or half-integer values.

- (0, 1/2) labels the positive-chirality spinor representation, with spinor λ_a.
- (1/2, 0) labels the negative-chirality spinor representation, with spinor $\tilde{\lambda}_{\dot{a}}$.
- The vector representation of the Lorentz group is the (1/2, 1/2) representation.

Any 2×2 matrix has at most rank 2, so a momentum p^μ may be viewed as the product of Weyl spinors λ_a and μ_a in (0, 1/2) and spinors $\tilde{\lambda}_{\dot{a}}$ and $\tilde{\mu}_{\dot{a}}$ in (1/2, 0). In fact, we can write $p \cdot \overline{\sigma}$, with $\det(p \cdot \overline{\sigma}) = p^2$, as

$$p_{a\dot{a}} = (p \cdot \overline{\sigma})_{a\dot{a}} = p^0 \mathbb{1} + \vec{p} \cdot \vec{\sigma} = \lambda_a(p)\tilde{\lambda}^{\intercal}_{\dot{a}}(p) + \mu_a(p)\tilde{\mu}^{\intercal}_{\dot{a}}(p)\,. \tag{1.62}$$

If $\det(p \cdot \overline{\sigma}) = p^2 = 0$, the rank of $p \cdot \overline{\sigma}$ is less than 2. Any 2×2 matrix A whose rank is less than 2 can be written as $A = vw^{\intercal}$, with generic $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. In fact,

$$A = \begin{pmatrix} v_1 w_1 & v_1 w_2 \\ v_2 w_1 & v_2 w_2 \end{pmatrix}, \tag{1.63}$$

which implies that $\det(A) = 0$. In particular, we can write $p \cdot \overline{\sigma}$ in terms of $\lambda_a(p)$ and $\tilde{\lambda}_{\dot{a}}(p)$ only,

$$p_{a\dot{a}} = (p \cdot \overline{\sigma})_{a\dot{a}} = \lambda_a(p)\tilde{\lambda}^{\intercal}_{\dot{a}}(p) = \begin{pmatrix} p^+ & p_\perp^* \\ p_\perp & p^- \end{pmatrix} \tag{1.64}$$

with

$$\lambda_a(p) = \frac{1}{\sqrt{p^+}} \begin{pmatrix} p^+ \\ p_\perp \end{pmatrix}, \qquad \tilde{\lambda}_{\dot{a}}(p) = \frac{1}{\sqrt{p^+}} \begin{pmatrix} p^+ \\ p_\perp^* \end{pmatrix}. \tag{1.65}$$

Since $\det(p \cdot \overline{\sigma}) = 0$, its eigenvalues are $\mu_1 = 0$ and $\mu_2 = p^+ + p^- = 2E$. The eigenvector corresponding to μ_2 is $\lambda_a(p) = \frac{1}{\sqrt{p^+}} \begin{pmatrix} p^+ \\ p_\perp \end{pmatrix}$, with a suitable normalization. So $(p \cdot \overline{\sigma})_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ represents the projection onto the eigenvector associated to the non-null eigenvalue.

If the momentum $p_{a\dot{a}}$ is complex, it has three complex parameters, since there are four complex components and one mass-shell condition, $p^+ p^- = p_\perp p_\perp^*$. The counting can be done otherwise: the spinors λ_a and $\tilde{\lambda}_{\dot{a}}$ have each two complex parameters, but their product $p_{a\dot{a}}$ is invariant under

$$\lambda_a \to z\lambda_a\,, \qquad \tilde{\lambda}_{\dot{a}} \to z^{-1}\tilde{\lambda}_{\dot{a}}\,, \tag{1.66}$$

with z complex. For a complex momentum, λ_a and $\tilde{\lambda}_{\dot{a}}$ are independent. In order for the momentum p^μ to be real, $\tilde{\lambda}_{\dot{a}} = \lambda_a^\dagger$, and thus z must be a phase, $z = e^{i\theta}$.

The map (1.66), which keeps $p_{a\dot{a}}$ invariant, is a **little group** map. As we said in sec. 1.1, the little group is the subgroup of the homogeneous Lorentz group made of the Lorentz maps that leave p^μ invariant, $p^\mu = \Lambda^\mu_\nu p^\nu$. Under little group scaling,

$$\lambda_a \to e^{i\theta}\lambda_a, \qquad \tilde{\lambda}_{\dot{a}} \to e^{-i\theta}\tilde{\lambda}_{\dot{a}}\,, \tag{1.67}$$

the helicity operator,

$$h = \frac{1}{2}\left(\lambda_a \frac{\partial}{\partial\lambda_a} - \tilde{\lambda}_{\dot{a}}\frac{\partial}{\partial\tilde{\lambda}_{\dot{a}}}\right)\,, \tag{1.68}$$

is invariant. h assigns helicity $+1/2$ to λ_a and helicity -1/2 to $\tilde{\lambda}_{\dot{a}}$. We can then define the helicity on an n-point amplitude as

$$h = \frac{1}{2}\sum_{i=1}^{n}\left(\lambda_{ia} \frac{\partial}{\partial\lambda_{ia}} - \tilde{\lambda}_{i\dot{a}}\frac{\partial}{\partial\tilde{\lambda}_{i\dot{a}}}\right)\,. \tag{1.69}$$

Likewise, we can write $p\cdot\sigma$ as

$$p^{\dot{b}b} = (p\cdot\sigma)^{\dot{b}b} = \tilde{\lambda}^{\dot{b}}(p)\lambda^b(p) = \begin{pmatrix} p^- & -p_\perp^* \\ -p_\perp & p^+ \end{pmatrix}\,, \tag{1.70}$$

with

$$\lambda^b(p) = \frac{1}{\sqrt{p^+}}\begin{pmatrix} -p_\perp \\ p^+ \end{pmatrix}, \qquad \tilde{\lambda}^{\dot{b}}(p) = \frac{1}{\sqrt{p^+}}\begin{pmatrix} -p_\perp^* \\ p^+ \end{pmatrix}\,. \tag{1.71}$$

Comparing to the Weyl spinors $\xi_\pm$ introduced above, we see that

$$\begin{aligned} &\{\lambda_a(p) = \xi_+(p), \quad \lambda^a(p) = \xi_-^\dagger(p)\} \in (0,1/2)\,, \\ &\{\tilde{\lambda}^{\dot{a}}(p) = \xi_-(p), \quad \tilde{\lambda}_{\dot{a}}(p) = \xi_+^\dagger(p)\} \in (1/2,0)\,. \end{aligned} \tag{1.72}$$

In order to raise and lower indices, we can introduce anti-symmetric tensors,

$$\begin{aligned} \epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2\,, \\ \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2\,. \end{aligned} \tag{1.73}$$

such that

$$\lambda_a = \epsilon_{ab}\lambda^b\,, \quad \lambda^a = \epsilon^{ab}\lambda_b\,, \quad \tilde{\lambda}^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{b}}\,, \quad \tilde{\lambda}_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}^{\dot{b}}\,, \tag{1.74}$$

so ϵ^{ab} acts in the (0, 1/2), i.e. right-handed representation and $\epsilon^{\dot{a}\dot{b}}$ acts in the (1/2, 0), i.e. left-handed representation.

Note that

- $\lambda_a = \epsilon_{ab}\lambda^b$ is equivalent to $\xi_+ = i\sigma^2\xi_-^*$,
- $\tilde{\lambda}^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{b}}$ is equivalent to $\xi_- = -i\sigma^2\xi_+^*$.

The van-der-Waerden notation for the Weyl spinors λ in terms of dotted and undotted indices, is equivalent to the notation in terms of the Weyl spinor ξ. They are both used in the literature and we will shuttle from one to the other.

Thus, the spinor products can be written as

$$\begin{aligned}\langle kp\rangle &\equiv \xi_-^\dagger(k)\xi_+(p) = \lambda^a(k)\lambda_a(p) = \lambda^a(k)\epsilon_{ab}\lambda^b(p) = -\lambda_b(k)\lambda^b(p) = -\lambda^b(p)\lambda_b(k) = -\langle pk\rangle\,,\\ [kp] &\equiv \xi_+^\dagger(k)\xi_-(p) = \tilde{\lambda}_{\dot{a}}(k)\tilde{\lambda}^{\dot{a}}(p) = \tilde{\lambda}_{\dot{a}}(k)\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{b}}(p) = -\tilde{\lambda}^{\dot{b}}(k)\tilde{\lambda}_{\dot{b}}(p) = -\tilde{\lambda}_{\dot{b}}(p)\tilde{\lambda}^{\dot{b}}(k) = -[pk]\end{aligned} \tag{1.75}$$

where the antisymmetry is explicitly implemented through the ϵ^{ab} tensors, just like it was through $i\sigma^2$ for the Weyl spinors ξ, and the fact that the spinors are commuting objects.

In terms of λ spinors, the Gordon identity is (see app. H.3)

$$\begin{aligned}\left\langle p^+\right|\gamma^\mu\left|p^+\right\rangle &= \tilde{\lambda}_{\dot{a}}(p)(\sigma^\mu)^{\dot{a}a}\lambda_a(p) = 2p^\mu\,,\\ \left\langle p^-\right|\gamma^\mu\left|p^-\right\rangle &= \lambda^a(p)(\overline{\sigma}^\mu)_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(p) = 2p^\mu\,.\end{aligned} \tag{1.76}$$

In terms of λ spinors, the charge conjugation

$$\left\langle p^-\right|\gamma^\mu\left|q^-\right\rangle = \left\langle q^+\right|\gamma^\mu\left|p^+\right\rangle \tag{1.77}$$

i.e.

$$\xi_-^\dagger(p)\overline{\sigma}^\mu\xi_-(q) = \xi_+(q)\sigma^\mu\xi_+(p) \tag{1.78}$$

becomes

$$\lambda^a(p)(\overline{\sigma}^\mu)_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(q) = \tilde{\lambda}_{\dot{a}}(q)(\sigma^\mu)^{\dot{a}a}\lambda_a(p) \tag{1.79}$$

where we use the identity (see app. H.3)

$$(\sigma^\mu)^{\dot{a}a} = \epsilon^{\dot{a}\dot{b}}(\overline{\sigma}^\mu)^T_{\dot{b}b}\epsilon^{ba} \tag{1.80}$$

which is equivalent to $\sigma^2\overline{\sigma}^\mu\sigma^2 = (\sigma^\mu)^T$.

One can work out the squaring as a product of two currents and using the Gordon identity,

$$\begin{aligned}\frac{1}{2}\left\langle p^-\right|\gamma^\mu\left|p^-\right\rangle\left\langle q^+\right|\gamma_\mu\left|q^+\right\rangle &= \frac{1}{2}\lambda^a(p)(\overline{\sigma}^\mu)_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(p)\tilde{\lambda}_{\dot{b}}(q)(\sigma^\mu)^{\dot{b}b}\lambda_b(q) = 2p\cdot q\\ &= \lambda^a(p)\tilde{\lambda}^{\dot{a}}(p)\tilde{\lambda}_{\dot{a}}(q)\lambda_a(q)\\ &= \langle pq\rangle[qp]\,,\end{aligned} \tag{1.81}$$

where we used the Fierz identity for Pauli matrices $(\overline{\sigma}_\mu)_{a\dot{a}}(\sigma^\mu)^{\dot{b}b} = 2\delta^b_a\delta^{\dot{b}}_{\dot{a}}$ and the spinor products. Therefore,

$$\left\langle p^- \middle| \gamma^\mu \middle| p^- \right\rangle \left\langle q^+ \middle| \gamma_\mu \middle| q^+ \right\rangle = 2\langle pq\rangle[qp]\,. \tag{1.82}$$

1.3 $e^-e^+ \to \mu^-\mu^+$ scattering

We consider the amplitude for $e^-e^+ \to \mu^-\mu^+$ scattering at fixed helicities.

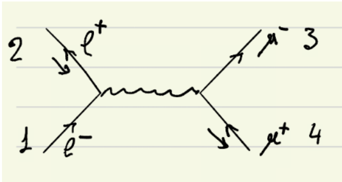


Figure 1.1: $e^-e^+ \to \mu^-\mu^+$ scattering.

We choose the momenta to be all outgoing, so that momentum conservation is $\sum_{i=1}^{4} p_i^\mu = 0$. Accordingly, helicites are for all outgoing momenta, i.e. an incoming left-handed electron is labelled as an outgoing right-handed positron, so e.g. the amplitude $e^-_L(-p_1)e^+_R(-p_2) \to \mu^-_R(p_3)\mu^+_L(p_4)$ becomes $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$, whose result is (see eq. H.4)

$$iM_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = i\frac{2e^2}{s_{12}}\langle 24\rangle[31]\,. \tag{1.83}$$

We can also write it in terms of right-handed spinor products only (see eq. H.4),

$$iM_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = -2ie^2\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle}\,. \tag{1.84}$$

or left-handed spinor products only,

$$iM_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = -2ie^2\frac{[13]^2}{[12][34]}\,. \tag{1.85}$$

The amplitude above is the beginning of an infinite tower of amplitudes, called maximally helicity violating (MHV) amplitudes, which are made of the four fermions above plus $(n-4)$ additional positive-helicity gluons or photons, i.e. whose helicity configuration is $(--\underbrace{+...+}_{n-2})$, and which can be written in terms of right-handed spinor products only. It is also the beginning of an infinite tower of $\overline{\text{MHV}}$ amplitudes made of the four fermions above plus $(n-4)$ negative-helicity gluons or photons, whose helicity configuration is $(++\underbrace{-...-}_{n-2})$, and which can be written in terms of left-handed spinor products only. Of course, only four-point amplitudes, with helicity configuration $(++--)$ can be both MHV and $\overline{\text{MHV}}$.

We have computed the amplitude $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$. four-point amplitudes have in principle

16 helicity configurations, however because helicity is conserved on massless fermion lines, $e^-e^+ \to \mu^-\mu^+$ has only four allowed helicity configurations. Further, they are all related by parity (P) and charge conjugation (C) on either or both of the fermion lines. C swaps a fermion with its anti-fermion. We can e.g. let C act on the muon line, $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^+}, 4^-_{\mu^-}) = M_4(1^+_{e^+}, 2^-_{e^-}, 4^-_{\mu^-}, 3^+_{\mu^+})$. Then we can swap the labels 3 and 4, which is equivalent to flip the helicities of particles 3 and 4. Thus we obtain

$$\begin{aligned} iM_4(1^+_{e^+}, 2^-_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+}) &= i\frac{2e^2}{s_{12}}\langle 23\rangle[41] \\ &= 2ie^2\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle} \\ &= 2ie^2\frac{[14]^2}{[12][34]} \, . \end{aligned} \tag{1.86}$$

Since for massless particles, helicity and chirality coincide, a parity transformation flips all helicities and conjugates spinors, thus, under P, $\langle pk\rangle \leftrightarrow [kp]$ (we will explain the sign later). Thus, we obtain

$$\begin{aligned} iM_4(1^-_{e^+}, 2^+_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+}) = i\frac{2e^2}{s_{12}}[24]\langle 31\rangle \, , \\ iM_4(1^-_{e^+}, 2^+_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = i\frac{2e^2}{s_{12}}[23]\langle 41\rangle \, . \end{aligned} \tag{1.87}$$

Finally, since $s_{13} = s_{24}$, we can also write

$$M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = e^{i\phi}2e^2\frac{s_{13}}{s_{12}} \, , \tag{1.88}$$

up to a phase ϕ. Choosing the momenta,

$$p_1 = (E, 0, 0, E), \qquad p_2 = (E, 0, 0, -E) \, , \qquad p_3 = (E, \vec{k}), \qquad p_4 = (E, -\vec{k}) \, , \tag{1.89}$$

with $|\vec{p_3}| = E$ and $\vec{p_3} \cdot \hat{z} = E\cos\theta$. Then

$$\begin{aligned} s_{12} &= (p_1 + p_2)^2 = 4E^2 \, , \\ s_{13} &= (p_1 - p_3)^2 = (p_2 - p_4)^2 = -2E^2(1 - \cos\theta) \, , \\ s_{14} &= (p_1 - p_4)^2 = (p_2 - p_3)^2 = -2E^2(1 + \cos\theta) \, . \end{aligned} \tag{1.90}$$

Therefore

$$M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = e^{i\phi}2e^2\frac{s_{13}}{s_{12}} = -e^{i\phi}2e^2\frac{1 - \cos\theta}{2} \, , \tag{1.91}$$

which vanishes in the forward limit $\theta \to 0$, since the total angular momentum S is not conserved in the beam direction (incoming S is -1 and the outgoing S is +1).

Likewise, we can write

$$M_4(1^+_{e^+}, 2^-_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+}) = e^{i\phi'} 2e^2 \frac{s_{14}}{s_{12}} = -e^{i\phi'} 2e^2 \frac{1 + cos\theta}{2} \,. \tag{1.92}$$

with a phase ϕ', which is maximised in the forward limit $\theta \to 0$, since S is conserved in the beam direction (incoming S is -1 and the outgoing S is -1).

Finally, the other two helicity amplitudes, $M_4(1^-_{e^+}, 2^+_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+})$ and $M_4(1^-_{e^+}, 2^+_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$, equal the two above, up to a phase.

The cross section for $e^-e^+ \to \mu^-\mu^+$ is proportional to the square of the amplitude. If the beam (initial-state) and final-state particles are not polarised, which is the default set up in collider physics, all the helicity configurations contribute to the amplitude. However, different helicity configurations do not interfere, so the square of the amplitude equals the sum of the squares of the helicity amplitudes,

$$\frac{d\sigma}{d\cos\theta}(e^-e^+ \to \mu^-\mu^+) \propto \sum_{hel} \left| M(e^-e^+ \to \mu^-\mu^+) \right|^2 . \tag{1.93}$$

with

$$\begin{aligned} \left| M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) \right|^2 &= \left| M_4(1^-_{e^+}, 2^+_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+}) \right|^2 = e^4(1 - \cos\theta)^2 \,, \\ \left| M_4(1^+_{e^+}, 2^-_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+}) \right|^2 &= \left| M_4(1^-_{e^+}, 2^+_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) \right|^2 = e^4(1 + \cos\theta)^2 \,. \end{aligned} \tag{1.94}$$

So, we have

$$\sum_{hel} \left| M(e^-e^+ \to \mu^-\mu^+) \right|^2 = 8e^4 \frac{s_{13}^2 + s_{14}^2}{s_{12}^2} = 4e^2(1 + \cos^2\theta) \,. \tag{1.95}$$

1.3.1 $e^-e^+ \to q\bar{q}$ scattering

Likewise, one can get the amplitudes for $e^-e^+ \to q\bar{q}$ scattering. We write the full amplitude as a colour-stripped amplitude times an overall factor containing couplings and colour factors.

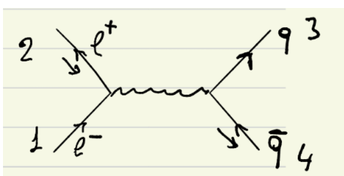


Figure 1.2: $e^-e^+ \to q\bar{q}$ scattering.

Since we normalise the colour matrices in the fundamental representation as $\mathrm{Tr}(\mathrm{t^a t^b}) = \mathrm{T_F}\delta^{ab}$, with $\mathrm{T_F} = 1$, we shall get a $1/\sqrt{2}$ factor in the Feynman rule for the quark-photon vertex, or the quark-gluon vertex, in the colour-stripped amplitude. To compensate for that, the coupling g will be replaced by $\sqrt{2}e$ for QED. So, we write the amplitude as

$$M_4(e^-e^+ \to q\bar{q}) = (\sqrt{2}e)^2 Q_q Q_e \delta_{i_s}^{i_4} A_4(e^-e^+ \to q\bar{q}) \,, \tag{1.96}$$

with Q_q and Q_e the quark and the electron electric charges, and with the colour-stripped A_4 amplitudes,

$$iA_4(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^-_{\bar{q}}) = i\frac{\langle 24\rangle[31]}{s_{12}} = i\frac{s_{13}}{s_{12}}e^{i\phi}\,, \tag{1.97}$$

$$iA_4(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_{\bar{q}}) = i\frac{\langle 23\rangle[41]}{s_{12}} = i\frac{s_{14}}{s_{12}}e^{i\phi'}\,. \tag{1.98}$$

1.3.2 $eq \to eq$ scattering

From the amplitudes for $e^-e^+ \to q\bar{q}$ scattering, we can obtain by crossing the amplitudes for $eq \to eq$ scattering, fig. 1.3, necessary to compute deep inelastic scattering (DIS).

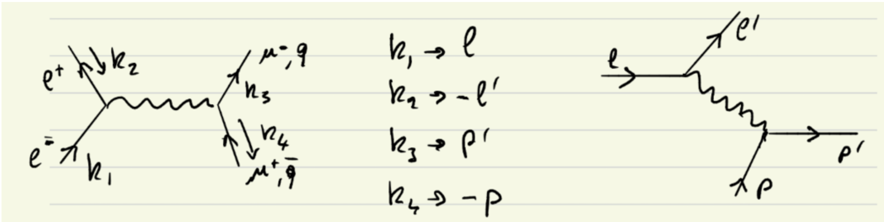


Figure 1.3: Crossing symmetry on $e^-e^+ \to q\bar{q}$ scattering.

The squared amplitude for $eq \to eq$ scattering, summed over helicities, is (see Exercise H.5)

$$\sum_{hel} |M|^2 = 8e^4 Q_q^2\,\frac{s^2+u^2}{t^2}\,, \tag{1.99}$$

where the s^2 term comes from the scattering of L(R)-handed electrons on L(R)-handed quarks, and the u^2 term from the scattering of L(R)-handed electrons on R(L)-handed quarks.

In the parton model, the DIS cross section is

$$\frac{d\sigma}{dxdy} = 2\pi\alpha^2\frac{S}{(Q^2)^2}[1+(1-y)^2]\sum_i x f_{i/P}(x) Q_i^2\,. \tag{1.100}$$

where is S is the hadron centre-of-mass energy, such that $s = xS$, $x = \dfrac{Q^2}{2p\cdot q}$ is the Bjorken x, y is the fractional energy loss of the outgoing lepton, and the sum is over the quark flavours. The two terms 1 and $(1-y)^2$ stem from the s^2 term and the u^2 term in eq. (1.99), respectively. Thus, we see how the two different helicity structures of $eq \to eq$ scattering yield the y structure of the DIS cross section.

1.3.3 $q\bar{q} \to \ell^-\ell^+$ scattering

Finally, by a trivial crossing of $e^-e^+ \to q\bar{q}$, (i.e. swapping initial and final states), one can obtain the amplitudes for the production of a lepton pair in the $q\bar{q}$ - annihilation, in the collision of two protons, which is called Drell-Yan scattering,

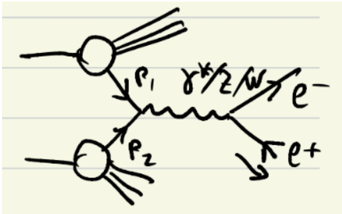


Figure 1.4: Drell-Yan scattering.

1.4 Polarisation vectors

We consider now the emission of gluons or photons in a helicity amplitude. As we know, the physical (or transverse) polarisation of a gluon or photon of momentum p and helicity $h = \pm$ is given by a 4-vector $\epsilon_h^\mu(p,k)$, with respect to an arbitrary null vector k^μ, $k^2 = 0$, not collinear to p, $p \cdot k \neq 0$, with the usual properties, (see app. H.6)

$$(\epsilon_\mu^\pm(p,k))^* = \epsilon_\mu^\mp(p,k) \qquad \text{and} \qquad \epsilon^\pm(p,k)\cdot p = 0\,, \tag{1.101}$$

normalised as

$$\epsilon^h(p,k)\cdot(\epsilon^{h'}(p,k))^* = -\delta^{hh'}\,, \tag{1.102}$$

and sum over polarizations,

$$\sum_h \epsilon_h^\mu(p,k)\epsilon_h^{\nu *}(p,k) = -g^{\mu\nu} + \frac{p^\mu k^\nu + p^\nu k^\mu}{p\cdot k}\,. \tag{1.103}$$

A polarisation vector satisfying these properties is

$$\epsilon_\mu^{\pm *}(p,k) = \pm\frac{\langle k^\mp|\,\gamma_\mu\,|p^\mp\rangle}{\sqrt{2}\,\langle k^\mp|\,|p^\pm\rangle}\,. \qquad \text{(see app. H.6)} \tag{1.104}$$

If we take the momentum p in the z-direction $p = (p,0,0,p)$ and the reference vector k as $k = (k,0,0,-k)$, so that p and k tag the light-cone directions, $p^+ = 2p$, $k^- = 2k$, the Weyl spinors are

$$\xi_+(p) = \begin{pmatrix}\sqrt{2p}\\ 0\end{pmatrix}, \qquad \xi_-(p) = \begin{pmatrix}0\\ \sqrt{2p}\end{pmatrix}, \qquad \xi_-(k) = \begin{pmatrix}-\sqrt{2k}\\ 0\end{pmatrix}, \tag{1.105}$$

Then the polarization vector is

$$\epsilon_\mu^{+*}(p,k) = \frac{\langle k^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\langle kp\rangle} = \frac{\xi_-^\dagger(k)\overline{\sigma}^\mu\xi_-(p)}{\sqrt{2}\xi_-^\dagger(k)\xi_+(p)} = -\frac{\sqrt{4kp}\;(0,1,-i,0)}{\sqrt{2}\sqrt{4kp}} = -\frac{1}{\sqrt{2}}\;(0,1,i,0)^*\,. \tag{1.106}$$

up to an overall sign, this coincides with the usual definition of right-handed polarisation vector.

Further, consider an azimuthal rotation by the angle ϕ around p. Then $\xi_-(p)$ in the numerator picks up a phase $e^{-i\phi/2}$, $\xi_+(p)$ in the denominator picks up a phase $e^{i\phi/2}$. Thus, $\epsilon_\mu^{+*}(p,k)$ gets the phase $e^{-i\phi}$, i.e. $\epsilon^+_\mu(p,k)$ transforms by $e^{i\phi}$, consistently with a right-handed spin-1 particle.

Finally, in 2×2 matrix notation,

$$\epsilon^{+*}_\mu(p,k) = \frac{\langle k^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\langle kp\rangle} = \frac{\lambda^a(k)(\overline{\sigma}_\mu)_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(p)}{\sqrt{2}\langle kp\rangle}\,. \tag{1.107}$$

Thus, using Fierz identity $\overline{\sigma}^{a\dot{a}}_\mu(\sigma^\mu)^{\dot{b}b} = 2\delta^b_a\delta^{\dot{b}}_{\dot{a}}$,

$$(\epsilon^{+*}(p,k))^{\dot{b}b} = (\epsilon^{+*}\cdot\sigma)^{\dot{b}b} = \frac{\lambda^a(k)(\overline{\sigma}_\mu)_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(p)}{\sqrt{2}\langle kp\rangle}(\sigma^\mu)^{\dot{b}b} = \sqrt{2}\frac{\lambda^b(k)\tilde{\lambda}^{\dot{b}}(p)}{\langle kp\rangle}\,. \tag{1.108}$$

Likewise,

$$(\epsilon^{-*}(p,k))_{a\dot{a}} = (\epsilon^{-*}\cdot\overline{\sigma})_{a\dot{a}} = -\sqrt{2}\frac{\tilde{\lambda}_{\dot{a}}(k)\lambda_a(p)}{[kp]}\,. \tag{1.109}$$

Choosing two different reference vectors yields

$$\begin{aligned}\epsilon^{+*}(p,k) - \epsilon^{+*}(p,q) &= \frac{\langle k^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\langle kp\rangle} - \frac{\langle q^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\langle qp\rangle}\\ &= \frac{-\langle k^-|\,\gamma_\mu\,|p^-\rangle\,\langle pq\rangle + \langle q^-|\,\gamma_\mu\,|p^-\rangle\,\langle pk\rangle}{\sqrt{2}\langle kp\rangle\langle qp\rangle}\,.\end{aligned} \tag{1.110}$$

Using $\not{p} = |p^-\rangle\,\langle p^-| + |p^+\rangle\,\langle p^+|$, where $|p^+\rangle\,\langle p^+|$ does not contribute in this case, we obtain

$$\begin{aligned}\epsilon^{+*}(p,k) - \epsilon^{+*}(p,q) &= \frac{-\langle k^-|\,\gamma_\mu\not{p}\,|q^+\rangle + \langle q^-|\,\gamma_\mu\not{p}\,|k^+\rangle}{\sqrt{2}\langle kp\rangle\langle qp\rangle}\\ &= \frac{\langle q^-|\,\not{p}\gamma_\mu + \gamma_\mu\not{p}\,|k^+\rangle}{\sqrt{2}\langle kp\rangle\langle qp\rangle}\\ &= 2p_\mu\frac{\langle qk\rangle}{\sqrt{2}\langle kp\rangle\langle qp\rangle}\,,\end{aligned} \tag{1.111}$$

where we have applied charge conjugation on the first current (see app. H.9). We have found that

$$\epsilon^+_\mu(p,k) = \epsilon^+_\mu(p,q) + p_\mu f(k,q,p)\,. \tag{1.112}$$

In a gauge-invariant amplitude, with on-shell gluons and/or photons, replacing the polarisation of a gluon or photon by its momentum yields zero by the Ward identity,

$$A = A^\mu(p)\epsilon_\mu(p) \to A^\mu(p)p_\mu = 0\,. \tag{1.113}$$

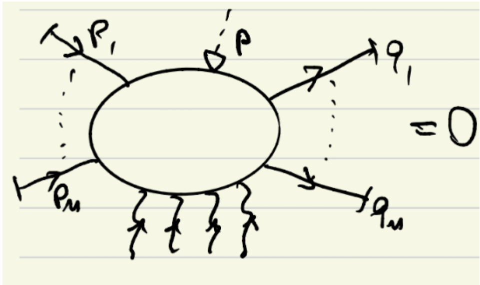


Figure 1.5: Ward identity.

i.e. the amplitude is invariant under $\epsilon_\mu(p) \to \epsilon_\mu(p) + p_\mu f(p)$. The change of the polarisation vector between two different reference vectors (1.111) is of this type, so we conclude that in a gauge-invariant amplitude or in fact in any gauge-invariant subset of an amplitude, we can choose the reference vector as we like. The outcome will not depend on that choice.

An additional property, which is useful when computing amplitudes with external fermions, is the contraction of the polarisation vector with the γ matrix,

$$\gamma \cdot \epsilon^{\pm *}(p,k) = \pm \frac{\sqrt{2}}{\langle k^{\mp} | p^{\pm} \rangle} \left(\left| p^{\mp} \right\rangle \left\langle k^{\mp} \right| + \left| k^{\pm} \right\rangle \left\langle p^{\pm} \right| \right) . \tag{1.114}$$

which is shown in app. H.8.

An educated choice of the reference vectors can then simplify dramatically a computation. To that effect, the following identities hold (see app. H.8),

$$k \cdot \epsilon^{\pm}(p,k) = 0 \,, \tag{1.115}$$

$$\epsilon^{\pm}(p_i,k) \cdot \epsilon^{\pm}(p_j,k) = 0 \,, \tag{1.116}$$

$$\epsilon^{+*}(p_i,p_j) \cdot \epsilon^{-*}(p_j,k) = 0 \,, \tag{1.117}$$

$$\epsilon^{+*}(p_i,k) \cdot \epsilon^{-*}(p_j,p_i) = 0 \,, \tag{1.118}$$

$$\not{\epsilon}^{\pm *}(p_i,p_j) \left| p_j^{\pm} \right\rangle = 0 \,, \tag{1.119}$$

$$\left\langle p_j^{\mp} \right| \not{\epsilon}^{\pm *}(p_i,p_j) = 0 \,. \tag{1.120}$$

Eq. (1.116) implies that it is convenient to choose the reference vectors of like-helicity gluons or photons to be the same, and to equal (eqs. (1.117) and (1.118)) the momentum of one of the opposite-helicity gluons/photons.

Finally, let us consider a parity transformation P of the polarisation vector (1.104). By flipping chiralities, $\epsilon^{\pm *}$ maps to $-\epsilon^{\mp *}$. So, the bookkeeping for P is to flip helicities and conjugate spinors by the map $\langle pk \rangle \to [pk]$ and multiply by -1 for each external gluon or photon. Since a scattering with only fermion lines and no external gluons or photons has an even number of spinor products, and for each external gluon or photon we add we should multiply by -1, a more concise bookkeeping for P is to flip all helicites and conjugate spinors by the map $\langle pk \rangle \to [kp]$.

1.4.1 $e^-e^+ \to \gamma\gamma$ scattering

In app. H.10, we apply the identities for the polarisation vectors above to compute the amplitude, $e_L^-(-p_1)e_R^+(-p_2) \to \gamma(p_3)\gamma(p_4)$. The outcome is

$$\begin{aligned} M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_\gamma, 4^-_\gamma) &= -2e^2 \frac{\langle 24\rangle^2}{\langle 13\rangle\langle 23\rangle}\,, \\ M_4(1^+_{e^+}, 2^-_{e^-}, 3^\pm_\gamma, 4^\pm_\gamma) &= 0\,. \end{aligned} \tag{1.121}$$

These are examples of general features: The former is a MHV amplitude, and is written in terms of right-handed spinor products only. The latter has only one negative (or only one positive) helicity, and vanishes.

By Bose symmetry, $M_4(1^+_{e^+}, 2^-_{e^-}, 3^-_\gamma, 4^+_\gamma)$ is obtained from $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_\gamma, 4^-_\gamma)$ by swapping labels 3 and 4. Therefore,

$$M_4(1^+_{e^+}, 2^-_{e^-}, 3^-_\gamma, 4^+_\gamma) = -2e^2 \frac{\langle 23\rangle^2}{\langle 14\rangle\langle 24\rangle}\,. \tag{1.122}$$

The other two amplitudes are obtained by P,

$$\begin{aligned} M_4(1^-_{e^+}, 2^+_{e^-}, 3^-_\gamma, 4^+_\gamma) &= -2e^2 \frac{[24]^2}{[13][23]}\,, \\ M_4(1^-_{e^+}, 2^+_{e^-}, 3^+_\gamma, 4^-_\gamma) &= -2e^2 \frac{[23]^2}{[14][24]}\,. \end{aligned} \tag{1.123}$$

Squaring and summing over helicities, we get

$$\sum_{hel} \left| M(e^-e^+ \to \gamma\gamma \right|^2 = 8e^4 \frac{s_{13}^2 + s_{14}^2}{s_{13}s_{14}} = 16e^4 \frac{1+\cos^2\theta}{1-\cos^2\theta}\,. \tag{1.124}$$

The squared amplitude diverges as $1/t$ in the forward limit, $\theta \to 0$, which is the expected behaviour for quark exchange in the t channel.

1.5 Infrared limits

1.5.1 $e^-e^+ \to q\bar{q}g$ scattering

In app. H.11, we compute the amplitude for the production of a gluon and a $q\bar{q}$ pair in e^+e^- annihilation $e_L^-(-p_1)e_R^+(-p_2) \to q_R(p_3)g_R(p_4)\bar{q}_L(p_5)$, which is relevant for 3-jet production in e^+e^- annihilation, and in the crossed channels for radiation emission in Drell-Yan production and in DIS. For the helicity configuration,

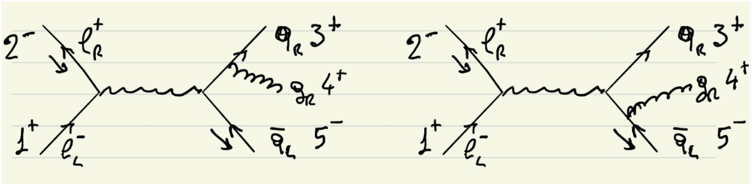


Figure 1.6: $e_L^-(-p_1)e_R^+(-p_2) \to q_R(p_3)g_R(p_4)\bar{q}_L(p_5)$ scattering.

the result is

$$M_5(e^-e^+ \to q\bar{q}g) = (\sqrt{2}e)^2 Q_q Q_e g (T^{Q_4})_{i_3}^{i_5} A_5(e^-e^+ \to q\bar{q}g)\,, \tag{1.125}$$

with

$$i)\quad A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) = -\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle}\,. \tag{1.126}$$

The amplitude with the negative-helicity gluon can be obtained by charge conjugation C and parity P. Let C act on the electron and the quark lines, $A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) = A_5(1^+_{e^-}, 2^-_{e^+}, 3^+_{\bar{q}}, 4^+_g, 5^-_q) = A_5(2^-_{e^+}, 1^+_{e^-}, 5^-_q, 4^+_g, 3^+_{\bar{q}})$. Then we can swap the labels $1 \leftrightarrow 2$ and $3 \leftrightarrow 5$, which is equivalent to flip the helicities of $1 \leftrightarrow 2$ and $3 \leftrightarrow 5$,

$$A_5(1^-_{e^+}, 2^+_{e^-}, 3^-_q, 4^+_g, 5^+_{\bar{q}}) = -\frac{\langle 13\rangle^2}{\langle 21\rangle\langle 54\rangle\langle 43\rangle}\,. \tag{1.127}$$

Then we apply parity P,

$$ii)\quad A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^-_g, 5^-_{\bar{q}}) = -\frac{[13]^2}{[12][34][45]}\,. \tag{1.128}$$

Note that by charge conjugation on the quark line only, i.e. by swapping labels 3 and 5 on eq. (1.126), we get the amplitude

$$\begin{aligned} iii)\quad A_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_g, 5^+_{\bar{q}}) &= -\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 54\rangle\langle 43\rangle} \\ &= -\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle}\,, \end{aligned} \tag{1.129}$$

while by charge conjugation on the electron line only, i.e. by swapping labels 1 and 2 on eq. (1.126), we get the amplitude

$$A_5(1^-_{e^+}, 2^+_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) = \frac{\langle 15\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle}\,, \tag{1.130}$$

then applying P, we get

$$iv)\quad A_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^-_g, 5^+_{\bar{q}}) = -\frac{[15]^2}{[12][34][45]}\,. \tag{1.131}$$

Thus, we have computed the 4 different helicity configurations of the final state by actually computing only one,

$$A_5(1^+_{e^+},2^-_{e^-},3^+_q,4^+_g,5^-_{\bar{q}}), \qquad A_5(1^+_{e^+},2^-_{e^-},3^+_q,4^-_g,5^-_{\bar{q}})\,,$$
$$A_5(1^+_{e^+},2^-_{e^-},3^-_q,4^+_g,5^+_{\bar{q}}), \qquad A_5(1^+_{e^+},2^-_{e^-},3^-_q,4^-_g,5^+_{\bar{q}})\,.$$

We will use their explicit values in order to compute the soft and collinear limits.

1.5.2 Soft Limit

Note that eq. (1.126) can be written as

$$i)\quad A_5(1^+_{e^+},2^-_{e^-},3^+_q,4^+_g,5^-_{\bar{q}}) = -\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 35\rangle}\cdot\frac{\langle 35\rangle}{\langle 34\rangle\langle 45\rangle}\,. \tag{1.132}$$

In the limit that the gluon becomes soft $p_4 \to 0$, the amplitude is singular, it goes like $1/p_4$ and this product can be interpreted as the non-radiative amplitude that we computed in eqs. (1.84), (1.85) and (1.97),

$$A_4(1^+_{e^+},2^-_{e^-},3^+_q,5^-_{\bar{q}}) = -\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 35\rangle} = -\frac{[13]^2}{[12][35]}\,, \tag{1.133}$$

times an eikonal factor,

$$S(3^+,4^+,5^-) = \frac{\langle 35\rangle}{\langle 34\rangle\langle 45\rangle}\,. \tag{1.134}$$

The amplitude with the negative-helicity gluon on the same quark line (1.128) can be written as

$$\begin{aligned} ii)\quad A_5(1^+_{e^+},2^-_{e^-},3^+_q,4^-_g,5^-_{\bar{q}}) &= -\frac{[13]^2}{[12][35]}\cdot\frac{[35]}{[34][45]} \\ &= A_4(1^+_{e^+},2^-_{e^-},3^+_q,5^-_{\bar{q}})\cdot S(3^+,4^-,5^-)\,. \end{aligned} \tag{1.135}$$

with

$$S(3^+,4^-,5^-) = \frac{[35]}{[34][45]}\,. \tag{1.136}$$

Likewise, using eqs. (1.86) and (1.98), the amplitudes we computed in eqs. (1.129) and (1.131) can be written as

$$\begin{aligned} iii)\quad A_5(1^+_{e^+},2^-_{e^-},3^-_q,4^+_g,5^+_{\bar{q}}) &= -\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 35\rangle}\cdot\frac{\langle 35\rangle}{\langle 34\rangle\langle 45\rangle} \\ &= A_4(1^+_{e^+},2^-_{e^-},3^-_q,5^+_{\bar{q}})\cdot S(3^-,4^+,5^+)\,. \end{aligned} \tag{1.137}$$

with

$$S(3^-,4^+,5^+) = -\frac{\langle 35\rangle}{\langle 34\rangle\langle 45\rangle} = -S(3^+,4^+,5^-)\,. \tag{1.138}$$

and

$$\text{iv)}\quad A_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^-_g, 5^+_{\bar{q}}) \;=\; -\frac{[15]^2}{[12][35]} \cdot \frac{[35]}{[34][45]}$$
$$= A_4(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 5^+_{\bar{q}}) \cdot S(3^-, 4^-, 5^+)\,. \tag{1.139}$$

with

$$S(3^-, 4^-, 5^+) = -\frac{[35]}{[34][45]} = -S(3^+, 4^-, 5^-)\,. \tag{1.140}$$

Apart from a sign, which is consistent with charge conjugation, i.e. swapping of labels 3 and 5, or otherwise with reflection identity, that we will define later, we see that the eikonal factor is independent of the helicities of 3 and 5.

These are general features. The soft-gluon limit of an n-point tree amplitude is

$$A_n^{tree}(1, ..., p, s, q, ..., n) = S(p, s^{\pm}, q) A_{n-1}^{tree}(1, ..., p, q, ..., n)\,, \tag{1.141}$$

where the eikonal factors can be taken to be

$$S(p, s^+, q) = \frac{\langle pq\rangle}{\langle ps\rangle\langle sq\rangle} \qquad \text{for the positive helicity gluon}\,, \tag{1.142}$$

$$S(p, s^-, q) = -\frac{[pq]}{[ps][sq]} \qquad \text{for the negative helicity gluon}\,, \tag{1.143}$$

where we have taken $S(p, s^-, q)$ with a minus sign because parity maps $S(p, s^+, q)$ to $S(p, s^-, q)$.

The eikonal factor is *universal*: it does not depend on the parton flavour (be quarks or gluons) or the spin of the emitters p and q (in fact, not even on the magnitude of the momenta p and q, since the eikonal factor is degree-zero in p and q; it depends only on their directions). The spin independence arises from the *classical* nature of the soft gluon: since its momentum vanishes its wavelength is very long, and it cannot resolve the details of the scattering process which has emitted it.

In QCD, the soft emission factorises, e.g. the emission of soft gluons from the quark-photon vertex is

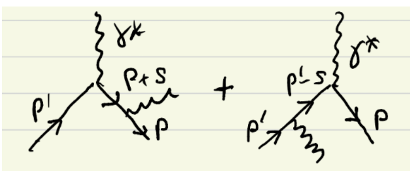


Figure 1.7: Gluon emission from quark-photon vertex.

$$iM_{soft} = -g\left(\frac{p\cdot\epsilon(s)}{p\cdot s} - \frac{p^{'}\cdot\epsilon(s)}{p^{'}\cdot s}\right)\overline{u}(p)t^a(-ie\Gamma^{\mu})u(p^{'})\,, \tag{1.144}$$

where Γ^{μ} represents the quark-photn interaction. Let us check that this is precisely what we are getting. Suppose the gluon of momentum s has positive helicity, and the reference vector is q,

$$\begin{aligned}
\frac{p\cdot\epsilon^+(s)}{p\cdot s}-\frac{p'\cdot\epsilon^+(s)}{p'\cdot s} = \quad & \frac{\langle q^-|\not{p}|s^-\rangle}{p\cdot s\sqrt{2}\langle qs\rangle}-\frac{\langle q^-|\not{p}'|s^-\rangle}{p'\cdot s\sqrt{2}\langle qs\rangle} \\
&= \frac{\sqrt{2}\langle qp\rangle[ps]}{\langle sp\rangle[ps]\langle qs\rangle}-\frac{\sqrt{2}\langle qp'\rangle[p's]}{\langle sp'\rangle[p's]\langle qs\rangle} \\
&= \frac{\sqrt{2}}{\langle qs\rangle}\frac{\langle qp\rangle\langle sp'\rangle-\langle qp'\rangle\langle sp\rangle}{\langle sp\rangle\langle sp'\rangle} \\
\text{use Schouten identity} \quad &= \frac{\sqrt{2}}{\langle qs\rangle}\frac{\langle qs\rangle\langle p'p\rangle}{\langle sp\rangle\langle sp'\rangle} \\
&= \sqrt{2}\frac{\langle pp'\rangle}{\langle ps\rangle\langle sp'\rangle} \qquad (1.145)
\end{aligned}$$

which is, up to the $\sqrt{2}$, our eikonal factor.

1.5.3 Collinear limit

Next, we consider the limit as gluon 4 becomes collinear to the quark 3, $p_3//p_4$. This limit is also singular, since the momentum $P = p_3+p_4$, of the quark before emitting the gluon, i.e. of the "quark parent", is also going on shell, $P^2 = 2p_3\cdot p_4 \to 0$. We may parametrize p_3 and p_4 as $p_3 = zP$ and $p_4 = (1-z)P$. Then using eq. (1.126), we have

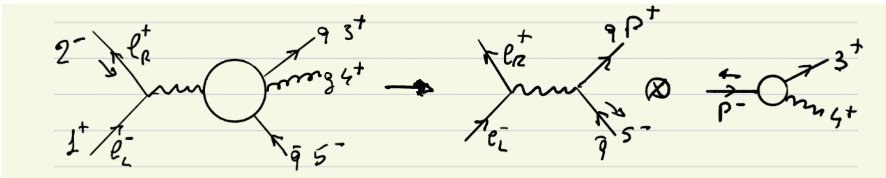


Figure 1.8: Collinear limit in $e^+e^- \to q\bar{q}g$ scattering.

$$i) \quad \lim_{p_3||p_4} A_5(1^+_{e^+},2^-_{e^-},3^+_q,4^+_g,5^-_{\bar{q}}) = -\frac{\langle 25\rangle^2}{\langle 12\rangle\langle P5\rangle}\frac{1}{\sqrt{1-z}\langle 34\rangle}\,, \qquad (1.146)$$

since the spinor product scales like the square root of the momentum. The five-point amplitude has factorised in the non-radiative amplitude (1.133),

$$A_4(1^+_{e^+},2^-_{e^-},P^+_q,5^-_{\bar{q}}) = -\frac{\langle 25\rangle^2}{\langle 12\rangle\langle P5\rangle} = -\frac{[1P]^2}{[12][P5]}\,, \qquad (1.147)$$

times the splitting amplitude (note the opposite helicites for leg P),

$$\text{Split}_-(3^+_q,4^+_g) = \frac{1}{\sqrt{1-z}\langle 34\rangle}\,. \qquad (1.148)$$

In order to determine the splitting amplitude for the negative-helicity gluon, we take eq. (1.128) with $p_3 = zP$ and $p_4 = (1-z)P$. Then

$$\begin{aligned} ii) \quad \lim_{p_3||p_4} A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^-_g, 5^-_{\bar{q}}) &= -\frac{[1P]^2}{[12][P5]} \frac{z}{\sqrt{1-z}[34]} \\ &= A_4(1^+_{e^+}, 2^-_{e^-}, P^+_q, 5^-_{\bar{q}}) \mathrm{Split}_-(3^+_q, 4^-_g) \,. \end{aligned} \tag{1.149}$$

with

$$\mathrm{Split}_-(3^+_q, 4^-_g) = \frac{z}{\sqrt{1-z}[34]} \,. \tag{1.150}$$

Likewise, we can take the collinear limit of the other two amplitudes we computed in eqs. (1.129) and (1.131) which have opposite helicities on the quark line,

$$iii) \quad \lim_{p_3||p_4} A_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_g, 5^+_{\bar{q}}) = -\frac{\langle 2P\rangle^2}{\langle 12\rangle\langle P5\rangle} \frac{z}{\sqrt{1-z}\langle 34\rangle} \,, \tag{1.151}$$

which can be written as the non-radiative amplitude out of eqs. (1.86) and (1.98),

$$A_4(1^+_{e^+}, 2^-_{e^-}, P^-_q, 5^+_{\bar{q}}) = \frac{\langle 2P\rangle^2}{\langle 12\rangle\langle P5\rangle} = \frac{[15]^2}{[12][P5]} \,, \tag{1.152}$$

times the splitting amplitude,

$$\mathrm{Split}_+(3^-_q, 4^+_g) = -\frac{z}{\sqrt{1-z}\langle 34\rangle} \,, \tag{1.153}$$

which is the parity conjugate of $\mathrm{Split}_-(3^+_q, 4^-_g)$.

$$\begin{aligned} iv) \quad \lim_{p_3||p_4} A_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^-_g, 5^+_{\bar{q}}) &= -\frac{[15]^2}{[12][P5]} \frac{1}{\sqrt{1-z}[34]} \\ &= A_4(1^+_{e^+}, 2^-_{e^-}, P^-_q, 5^+_{\bar{q}}) \mathrm{Split}_+(3^-_q, 4^-_g) \,, \end{aligned} \tag{1.154}$$

with

$$\mathrm{Split}_+(3^-_q, 4^-_g) = -\frac{1}{\sqrt{1-z}[34]} \,, \tag{1.155}$$

which is the parity conjugate of $\mathrm{Split}_-(3^+_q, 4^+_g)$.

These are also general features: the collinear limit of an n-point tree amplitude is

$$\lim_{k||q} A_n^{tree}(1, ..., k^{\lambda_k}, q^{\lambda_q}, ..., n) = \sum_{\lambda_P=\pm} \mathrm{Split}_{-\lambda_p}(k^{\lambda_k}, q^{\lambda_q}) A_n^{tree}(1, ..., P^{\lambda_P}, ..., n) \,. \tag{1.156}$$

The splitting amplitude depends on the nature and the spin of partons k and q. It features a sum over the helicity of the parent P, with the convention that the helicity of P is reversed between the $(n-1)$-point amplitude and the splitting amplitude, since helicites are taken as all outgoing on (splitting) amplitudes. However, in the case of a quark parent P, like in our specific case above, helicity conservation on the quark line implies that $\lambda_3 = \lambda_P$, i.e. only one helicity of P survives in the sum. Further, in QCD the parton flavour of the parent P is uniquely determined by the flavours of the collinear particles $q \to qg$, $\bar{q} \to \bar{q}g$, $g \to q\bar{q}$, $g \to gg$.

The splitting amplitudes are basically the square roots of the polarised Altarelli-Parisi splitting functions which appear in the DGLAP evolution equations. In particular, using the four splitting amplitudes we computed above, we can find the z dependence of the unpolarised $P_{qq}(z)$ splitting function,

$$\begin{aligned} P_{qq}(z) &\propto \left|\text{Split}_-(3_q^+, 4_g^+)\right|^2 + \left|\text{Split}_-(3_q^+, 4_g^-)\right|^2 + \left|\text{Split}_+(3_q^-, 4_g^+)\right|^2 + \left|\text{Split}_+(3_q^-, 4_g^-)\right|^2 \\ &\propto \frac{1}{1-z} + \frac{z^2}{1-z} = \frac{1+z^2}{1-z} \,. \end{aligned} \tag{1.157}$$

Including the $\delta(1-z)$ term from virtual gluons, and the + distribution to deal with the soft divergence as $z \to 1$, the complete $P_{qq}(z)$ is

$$P_{qq}(z) = C_F \left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right) . \tag{1.158}$$

The splitting amplitude $\text{Split}(k, q)$ is proportional to either $1/\langle kq \rangle$ or to $1/[kp]$, i.e. we can say in general that $\text{Split}(k, q) \propto e^{i\phi}/\sqrt{s_{kq}}$, where ϕ entails a phase factor. Note however that had we computed the splitting amplitude in a scalar theory, like ϕ^3, we would have obtained $\text{Split}(k, q) \propto 1/s_{kq}$.

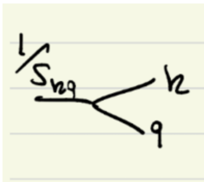


Figure 1.9: Split amplitude in ϕ^3 theory.

In the case of a massless gauge theory, like QCD, the collinear singularity is softened because helicity, or angular momentum, is not conserved in the splitting process. In fact, in the quark-gluon splitting,

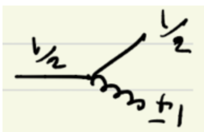


Figure 1.10: Quark-gluon splitting.

the initial helicity is 1/2 and the final helicity is -1/2, 3/2 or the initial helicity is -1/2 and the final one is -3/2, 1/2. Likewise, in a gluon splitting

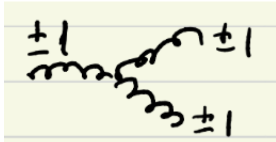


Figure 1.11: Gluon-gluon splitting.

the initial helicity is ±1, and the final helicity is 2,0,-2.

The splitting function P, which is proportional to $|\text{Split}|^2$, goes like $1/s_{kq}$. After phase-space integration, that leads to logarithmic collinear divergences. In a scalar theory, the splitting function goes like $1/s_{kq}^2$, leading to power-like collinear divergences.

1.6 $q\bar{q} \to (n-2)$ gluon scattering

1.6.1 Colour decomposition

For QCD, or $SU(N_c)$ gauge theories, helicity amplitudes have another great advantage: one can organise the amplitude as a linear combination of colour factor times partial (a.k.a. colour-ordered, colour-stripped, or sub-) amplitudes. Linear independence and the gauge invariance of the whole amplitude imply that the colour-stripped amplitudes are gauge invariant.

In the fundamental representation of $SU(N_c)$, the algebraic structure is given by the relation $[t^a, t^b] = if^{abc}t^c$, where the traceless hermitian $N_c \times N_c$ generator matrices $(t^a)_i^{\,j}$ carry quark indices $i, j = 1, .., N_c$ and a gluon index $a = 1, ..., N_c^2 - 1$. The t^a's are usually normalised by $\mathrm{Tr(t^a t^b)} = T_F \delta^{ab}$, with $T_F = 1/2$. Firstly, in order to avoid, a proliferation of $\sqrt{2}$'s in the sub-amplitudes, we rescale the t's, $T^a = \sqrt{2}t^a$, such that $\mathrm{Tr(T^a T^b)} = \delta^{ab}$. Thus, the Feynman rule for the quark-gluon vertex changes as

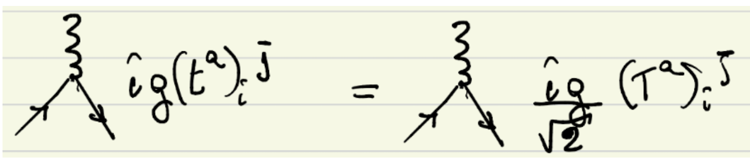
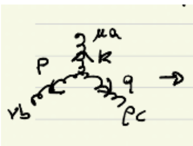


Figure 1.12: Feynman rule for the quark-gluon vertex.

In terms of T's, the $SU(N_c)$ algebra is given by $[T^a, T^b] = i\sqrt{2}f^{abc}T^c$, i.e. such that the structure constants f^{abc} are given by

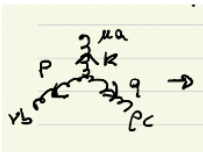
$$f^{abc} = -\frac{i}{\sqrt{2}}\mathrm{Tr}\Big([T^a, T^b]\, T^c\Big)\,. \tag{1.159}$$

The f^{abc}'s appear in the Feynman rules for the 3-gluon and 4-gluon vertices. The idea is to replace systematically in every Feynman diagram all the f^{abc}'s in terms of T^a's, using the relation above. For example, the Feynman rule for the three-gluon vertex is



$$gf^{abc}[g^{\mu\rho}(k-q)^{\nu}+g^{\mu\nu}(p-k)^{\rho}+g^{\rho\nu}(q-p)^{\mu}]\,. \quad (1.160)$$

Using eq. (1.159), we can devise the colour-ordered Feynman rule



$$-\frac{ig}{\sqrt{2}}\,\mathrm{tr}\big(T^aT^bT^c\big)\big[g^{\mu\rho}(q-k)^{\nu}+g^{\mu\nu}(k-p)^{\rho}+g^{\rho\nu}(p-q)^{\mu}\big] \quad (1.161)$$

$+$ non-cyclic permutation

The simplest example we consider is $q\bar{q}\to gg$ scattering,

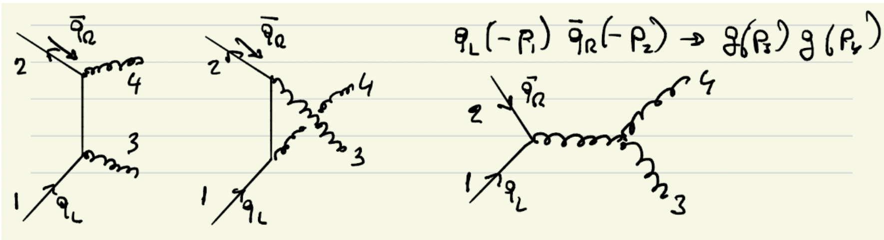


Figure 1.13: $q\bar{q}\to gg$ scattering

The first two (Abelian) diagrams yield $T^{a_3}T^{a_4}$ and $T^{a_4}T^{a_3}$ colour factors; the third (non-Abelian) diagram has a colour factor $f^{a_3a_4c}T^c$. One could replace $i\sqrt{2}f^{a_3a_4c}T^c=T^{a_3}T^{a_4}-T^{a_4}T^{a_3}$ (as e.g. done by Peskin in his lectures) and immediately get the $T^{a_3}T^{a_4}$ and $T^{a_4}T^{a_3}$ colour orders, which shows that the amplitude can be written in terms of sub-amplitudes as

$$M(1_{\bar{q}}^{\lambda_1},2_q^{-\lambda_1},3_g,4_g)=g^2[A(1_{\bar{q}}^{\lambda_1},2_q^{-\lambda_1},3_g,4_g)T^{a_3}T^{a_4}+A(1_{\bar{q}}^{\lambda_1},2_q^{-\lambda_1},4_g,3_g)T^{a_4}T^{a_3}]\,. \quad (1.162)$$

Although slightly longer in this case, a more general procedure is to use the colour-ordered Feynman rule for the 3-gluon vertex, and then use the Fierz identity for $SU(N_c)$ matrices,

$$(T^c)_{i_1}^{j_1}(T^c)_{i_2}^{j_2}=\delta_{i_1}^{j_2}\delta_{i_2}^{j_1}-\frac{1}{N_c}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\,, \quad (1.163)$$

whose ('t Hooft) graphical representation is

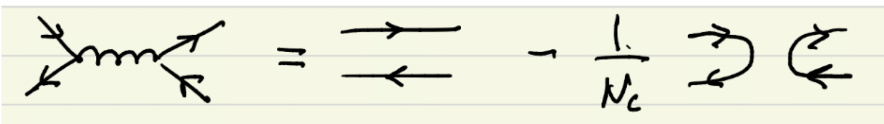
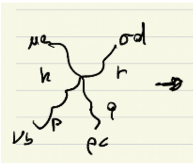


Figure 1.14: 't Hooft graphical representation.

Then the colour-ordered factor of the non-Abelian piece is

$$
\begin{aligned}
(T^c)^{j_2}_{i_1}[\mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^bT^aT^c)] &= (T^c)^{j_2}_{i_1}(T^c)^{j_4}_{i_3}\big((T^aT^b)^{i_3}_{j_4} - (T^bT^a)^{i_3}_{j_4}\big) \\
&= (T^aT^b)^{j_2}_{i_1} - (T^bT^a)^{j_2}_{i_1} - \cancelto{0}{\frac{1}{N_c}\mathrm{Tr}(T^aT^b - T^bT^a)}\delta^{j_2}_{i_1}\,, \qquad (1.164)
\end{aligned}
$$

and we obtain the amplitude in terms of colour-ordered amplitudes as above. This procedure can be used, though, for $q\bar{q} \to$ any number of gluons. For three or more gluons, we also need the four-gluon vertex,

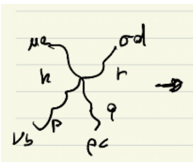


$$
\begin{aligned}
&-ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\,, \qquad (1.165)
\end{aligned}
$$

written in colour-ordered form. Each product of two f's can be unfolded by applying twice eq. (1.159). Then

$$
\begin{aligned}
f^{abe}f^{cde} &= -\frac{1}{2}\mathrm{Tr}\big([T^a, T^b]\,T^e\big)\mathrm{Tr}\big([T^c, T^d]\,T^e\big) \\
&= -\frac{1}{2}\Big(\mathrm{Tr}\big([T^a, T^b][T^c, T^d]\big) - \cancelto{0}{\frac{1}{N}\mathrm{Tr}\big([T^a, T^b]\big)\mathrm{Tr}\big([T^c, T^d]\big)}\Big)\,, \qquad (1.166)
\end{aligned}
$$

where in the second line we used the $SU(N_c)$ Fierz identity (1.163), so a product of traces can be written as a single trace. Unfolding the trace, and repeating the same procedure for $f^{ace}\, f^{bde}$ and $f^{ade}\, f^{bce}$, we obtain the colour-ordered Feynman rule of the four-gluon vertex,



$$
\frac{ig^2}{2}\,\mathrm{tr}\big[T^aT^bT^cT^d\big]\cdot(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma}) \qquad (1.167)
$$

$+\ 5$ non-cyclic permutations

Given the generic strings of T matrices, τ_1 and τ_2, using the $SU(N_c)$ Fierz identity (1.163), one obtains the following useful colour identities,

$$
\begin{aligned}
i)\quad \mathrm{Tr}(\tau_1T^c)\mathrm{Tr}(\tau_2T^c) &= (\tau_1)^{j_1}_{i_1}(T^c)^{i_1}_{j_1}(\tau_2)^{j_2}_{i_2}(T^c)^{i_2}_{j_2} \\
&= \mathrm{Tr}(\tau_1\tau_2) - \frac{1}{N_c}\mathrm{Tr}(\tau_1)\mathrm{Tr}(\tau_2)\,. \qquad (1.168) \\
ii)\quad \mathrm{Tr}(\tau_1T^c\tau_2T^c) &= (\tau_1)^{i_2}_{i_1}(T^c)^{i_3}_{i_2}(\tau_2)^{i_4}_{i_3}(T^c)^{i_1}_{i_4} \\
&= \mathrm{Tr}(\tau_1)\mathrm{Tr}(\tau_2) - \frac{1}{N_c}\mathrm{Tr}(\tau_1\tau_2)\,. \qquad (1.169)
\end{aligned}
$$

Finally, using $(T^cT^c)_{i_1}^{i_2} = C_f\delta_{i_1}^{i_2}$, where $C_f = \frac{N_c^2-1}{N_c}$, we have the identity

$$iii) \quad \mathrm{Tr}(\tau_1 T^c T^c \tau_2) = \frac{N_c^2-1}{N_c}\mathrm{Tr}(\tau_1\tau_2)\,. \tag{1.170}$$

However, in the two previous cases where we used the $SU(N_c)$. Fierz identity, eqs. (1.164) and (1.166), the $1/N_c$ term dropped. The $1/N_c$ term makes the $SU(N_c)$ generator matrices traceless. Since $U(N_c) = SU(N_c)\times U(1)$, the additional $U(1)$ generator is proportional to the identity,

$$(T_i^{a_{U(1)}})^j = \frac{1}{\sqrt{N_c}}\delta_i^j\,, \tag{1.171}$$

such that the $U(N_c)$ Fierz identity,

$$(T^c)_{i_1}^{j_2}(T^c)_{i_3}^{j_4} = \delta_{i_1}^{j_4}\delta_{i_3}^{j_2}\,, \tag{1.172}$$

differs from the $SU(N_c)$ Fierz identity (1.163) only by the $1/N_c$ term. The $U(1)$ generator (1.171) commutes with $SU(N_c)$, so it can be thought of as a photon, which is colourless and does not couple to gluons. That is why the $1/N_c$ term dropped in the usage of the $SU(N_c)$ Fierz identity above, and it will drop in all only-gluon amplitudes or in the ones for $q\bar{q} \to$ any number of gluons. It can survive only if a gluon is attached to quark lines on both ends. So it contributes only to the amplitudes with at least two quark lines. As we will see, the $U(1)$ generator can also be used to replace gluons with photons.

Using the colour-ordered Feynman rules of the 3-gluon and 4-gluon vertices, and the $SU(N_c)$ Fierz identity, it is possible to show that the general colour decomposition for the amplitude of $q\bar{q} \to$ any number of gluons can be reduced to a single string of T^a matrices,

$$M_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, 3_g, ..., n_g) = g^{n-2}\sum_{\sigma\in S_{n-2}}(T^{a\sigma_3}\cdots T^{a\sigma_n})_{i_1}^{j_2} A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, \sigma(3^{\lambda_3})\cdots\sigma(n^{\lambda_n}))\,, \tag{1.173}$$

where the sum is over the $(n-2)$ permutations of the gluon labels, and the λ's label helicities. The work is then all in computing the sub-amplitudes $A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, \sigma(3^{\lambda_3})\cdots\sigma(n^{\lambda_n}))$.

1.6.2 $q\bar{q} \to gg$ scattering

As we said above, for $n = 2$, the colour decomposition is given by eq. (1.162). In app. H.12, we compute the sub-amplitudes,

$$A_4^{tree}(1_{\bar{q}}^+, 2_q^-, 3_g^\pm, 4_g^\pm) = 0\,, \tag{1.174}$$

$$A_4^{tree}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-) = \frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{1.175}$$

$$A_4^{tree}(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+) = \frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \tag{1.176}$$

Eqs. (1.175) and (1.176) are MHV amplitudes for $q\bar{q} \to gg$. They can also be written as $\overline{\text{MHV}}$ amplitudes, i.e. in terms of left-handed spinor products [ ] only. Note that $A_4^{tree}(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-)$ and $A_4^{tree}(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+)$ are two distinct colour-ordered amplitudes, not related by Bose symmetry.

The non-vanishing helicity configurations are

$$M(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-) = g^2\big[\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}T^{a_3}T^{a_4} + \frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}T^{a_4}T^{a_3}\big]\,, \tag{1.177}$$

where we got the second term, $A(1_{\bar{q}}^+, 2_q^-, 4_g^-, 3_g^+)$, from $A(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+)$, eq. (1.176), by swapping labels 3 and 4. Likewise,

$$M(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+) = g^2\big[\frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}T^{a_3}T^{a_4} + \frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}T^{a_4}T^{a_3}\big]\,. \tag{1.178}$$

Note that, as colour-dressed amplitudes, $M(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-)$, eq. (1.177), and $M(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+)$, eq. (1.178), are related by Bose symmetry, by swapping labels 3 and 4.

By parity, we obtain the other two amplitudes,

$$\left|M(1_{\bar{q}}^-, 2_q^+, 3_g^-, 4_g^+)\right|^2 = \left|M(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-)\right|^2. \tag{1.179}$$

$$\left|M(1_{\bar{q}}^-, 2_q^+, 3_g^+, 4_g^-)\right|^2 = \left|M(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+)\right|^2. \tag{1.180}$$

Summing over helicities, the squared amplitude is

$$\sum_{hel} |M(1_{\bar{q}}, 2_q, 3_g, 4_g)|^2 = g^4\big[2\frac{(N_c^2-1)^2}{N_c}\frac{s_{13}^2+s_{14}^2}{s_{13}s_{14}} - 4N_c(N_c^2-1)\frac{s_{13}^2+s_{14}^2}{s_{12}^2}\big]\,, \tag{1.181}$$

(see app. H.13), or averaging over initial colours and helicities, i.e. dividing by $4N_c^2$, the averaged squared amplitude is

$$\sum_{hel} \left|\overline{M}(1_{\bar{q}}, 2_q, 3_g, 4_g)\right|^2 = g^4\big[\frac{(N_c^2-1)^2}{2N_c^3}\frac{s_{13}^2+s_{14}^2}{s_{13}s_{14}} - \frac{N_c^2-1}{N_c}\frac{s_{13}^2+s_{14}^2}{s_{12}^2}\big]\,. \tag{1.182}$$

The squared amplitude is symmetric under $s_{13} \leftrightarrow s_{14}$ or $t \leftrightarrow u$, exchange.

1.6.3 MHV amplitudes

Next, we consider the n-point amplitude for $q\bar{q} \to (n-2)$ gluons. Using the same identity as for two equal-helicity gluons (1.116), it is easy to show that

$$A_n^{tree}(1_{\bar{q}}^\pm, 2_q^\mp, 3_g^\pm, \ldots, n_g^\pm) = 0\,, \tag{1.183}$$

for all equal-helicity gluons. The two MHV amplitudes for $q\bar{q} \to gg$, eqs. (1.175) and (1.176), are extended to an arbitrary number of gluons. The amplitude for all positive-helicity gluons but one is

$$A_n^{tree}(1_{\bar{q}}^+, 2_q^-, 3_g^+, ..., i_g^-, ..., n_g^+) = \frac{\langle 2i\rangle^3 \langle 1i\rangle}{\langle 12\rangle\langle 23\rangle \cdots \langle (n-1)n\rangle\langle n1\rangle}\,, \tag{1.184}$$

where i labels the only negative-helicity gluon. Conversely, the amplitude for all negative-helicity gluons but one, is a $\overline{\text{MHV}}$ amplitude,

$$A_n^{tree}(1_{\bar{q}}^+, 2_q^-, 3_g^-, ..., i_g^+, ..., n_g^-) = (-1)^n \frac{[1i]^3[2i]}{[12][23] \cdots [(n-1)n][n1]}\,, \tag{1.185}$$

where i labels the only positive-helicity gluon.

It is apparent that MHV or $\overline{\text{MHV}}$ amplitudes for $q\bar{q} \to$ any number of gluons can have poles only in $s_{12}, s_{23}, ..., s_{n-1}, s_{n1}$. There are no poles in, say, s_{13} or s_{14}. Tree amplitudes have factorisation poles only when an intermediate state P goes on-shell,

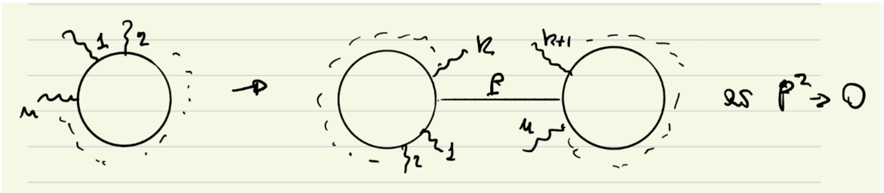


Figure 1.15: Multi-particle factorisation.

In a colour-ordered amplitude, particles must be cyclically adjacent in order to produce a pole. That is why poles in, say, s_{13} do not appear in $A(1_{\bar{q}}^{\pm}, 2_q^{\mp}, 3_g, ..., n_g)$, whatever are the gluon helicities. In particular, for MHV amplitudes, there are also no multi-particle poles, i.e. poles of the type $s_{12...m} = (k_1 + ... + k_m)^2$, with $m \geq 3$. We will explain this later.

1.7 Gluon amplitudes

1.7.1 Trace-based colour decomposition

Using the colour-ordered Feynman rules of the three-gluon and four-gluon vertices, and the $SU(N_c)$ Fierz identity to reduce any product of two traces to a single trace, it is possible to show that the general colour decomposition of an n-gluon tree amplitude can be reduced to a single trace of T^a matrices,

$$M_n^{tree}(1_g, \ldots, n_g) = g^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} \text{Tr}(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_n}}) A_n^{tree}(\sigma(1^{\lambda_1}) \cdots \sigma(n^{\lambda_n}))\,, \tag{1.186}$$

where the sum is over the $(n-1)!$ non-cyclic permutatons of the gluon labels. $\mathbb{Z}_n$ is the subset of the cyclic permutations, which leave the trace invariant.

Hence also the colour-ordered amplitudes:

i) $A_n^{tree}(\sigma(1^{\lambda_1})\cdots\sigma(n^{\lambda_n}))$ are cyclically invariant.

In order to get rid of the cyclic invariance in the colour decomposition (1.186), one could e.g. fix the position of gluon 1 and permute all the others,

$$M_n^{tree}(1,\ldots,n) = g^{n-2}\sum_{\sigma\in S_{n-1}} \text{Tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}})A_n^{tree}(1^{\lambda_1}\sigma(2^{\lambda_2})\cdots\sigma(n^{\lambda_n}))\,, \tag{1.187}$$

where we drop the gluon index, when dealing with gluons only.

Examining the Feynman diagrams which contribute to each sub-amplitude, one finds that sub-amplitude have a reflection identity,

$$ii) \qquad A_n^{tree}(n,\ldots,1) = (-1)^n A_n^{tree}(1,\ldots,n)\,. \tag{1.188}$$

Further, as we said in sec. 1.6.1, the $1/N_c$ term in the $SU(N_C)$ Fierz identity decouples in amplitudes with only gluons. Accordingly, the colour decomposition (1.186) is equally valid for $U(N_c) = SU(N_c)\times U(1)$. But gluon amplitudes which contain the $U(1)$ generator must vanish. So if we insert the $U(1)$ generator $(T^{a_{U(1)}})_i^{\,j} = \dfrac{1}{\sqrt{N_c}}\delta_i^j$ instead of T^{a_1} into the colour decomposition (1.186), and collect terms containing the same trace, we get a vanishing linear combination of $(n-1)$ sub-amplitudes,

$$A_n^{tree}(1,2,3,\ldots,n)+A_n^{tree}(2,1,3,\ldots,n)+A_n^{tree}(2,3,1,\ldots,n)+\cdots+A_n^{tree}(2,3,\ldots,1,n) = 0\,, \tag{1.189}$$

which we can also write as

$$iii) \qquad \sum_{\sigma\in\text{cyclic}} A_n^{tree}(1,\sigma(2),\ldots,\sigma(n)) = 0\,. \tag{1.190}$$

which is called photon decoupling identity.

The trace-based decomposition is clearly over-complete, the independent sub-amplitudes are less than $(n-1)!$. Let us see how many they are for $n = 4$. From the trace-based decomposition, we must consider the sub-amplitudes,

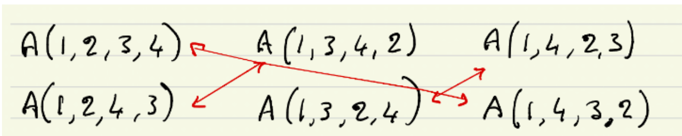


Figure 1.16: Reflection and cyclicity acting on four-gluon sub-amplitudes.

Reflection and cyclicity connect the sub-amplitudes according to the red arrows. Let us keep the ones of the first line. Further, let us apply the photon decoupling identity to gluon 1,

$$A(1,2,3,4) + A(1,4,2,3) + A(1,3,4,2) = 0\,. \tag{1.191}$$

So the independent sub-amplitudes are reduced to two, and using the photon decoupling identity on the other gluons, it is possible to see that one cannot reduce further the set of independent sub-amplitudes, which can be chosen to be any of the two not connected by reflection.

In fact, shortly after the trace-based decomposition was found, Kleiss and Kuijf (KK) found the relation [17],

$$A_n^{tree}(1,\{a\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in\{a\}\sqcup\!\sqcup\{\beta^T\}} A_n^{tree}\left(1,\sigma(\{a\},\{\beta^T\}),n\right) \,, \tag{1.192}$$

where $\{a\}\cup\{\beta\} = \{2,3,...,n-1\}$, n_β is the number of elements in $\{\beta\}$, $\{\beta^T\}$ is $\{\beta\}$ with the ordering reversed, and the shuffle $\{a\}\sqcup\!\sqcup\{\beta^T\}$ of the $(n-2)$ elements of $\{a\}\cup\{\beta^T\}$ are the permutations which preserve the ordering of the a_i within $\{a\}$ and of the β_i within $\{\beta^T\}$ while allowing for all possible orderings of the a_i with respect to β_i (like suffling two decks of cards). Note that the number of permutations given by the shuffle $\{a\}\sqcup\!\sqcup\{\beta^T\}$, i.e. the number of sub-amplitudes on the right-hand side of the KK relation, is given by the binomial coefficient $\begin{pmatrix} n-2 \\ n_\beta \end{pmatrix}$ with $n_\alpha + n_\beta = n-2$.

The KK relation includes the reflection and the photon-decoupling identities and reduces the number of independent sub-amplitudes to $(n-2)!$.

Let us apply the KK relation on the four-point amplitude,

- a) choose $\{\beta\} = \{\emptyset\}$, i.e. the null set, and $\{a\} = \{2,3\}$ then KK implies a trivial identity;
- b) choose $\{\beta\} = \{2,3\}$ and $\{a\} = \{\emptyset\}$, then KK implies

  $$A^{tree}(1,4,2,3) = A^{tree}(1,3,2,4) \overset{\text{cycl}}{=} A(3,2,4,1)\,, \tag{1.193}$$

  i.e. the reflection identity
- c) choose $\{a\} = \{2\}$ and $\{\beta\} = \{3\}$, then KK implies

  $$A^{tree}(1,2,4,3) = -A^{tree}(1,2,3,4) - A^{tree}(1,3,2,4)\,, \tag{1.194}$$

  i.e.

  $$A^{tree}(1,2,3,4) + A^{tree}(1,3,2,4) + A^{tree}(1,2,4,3) = 0\,, \tag{1.195}$$

  and with reflection and cyclicity,

  $$A^{tree}(1,2,3,4) + A^{tree}(1,4,2,3) + A^{tree}(1,3,4,2) = 0\,, \tag{1.196}$$

  i.e. the photon decoupling identity on gluon 1.

1.7.2 Multiperipheral colour decomposition

We shall consider now another colour decomposition. Let us rescale the structure constants (1.159) by setting,

$$F^{abc} = i\sqrt{2}f^{abc} = \mathrm{Tr}\big([T^a, T^b]T^c\big)\,. \tag{1.197}$$

Then the multiperipheral-based colour decomposition of the n-gluon tree amplitude is [18]

$$M_n^{tree}(1_g, \ldots, n_g) = g^{n-2} \sum_{\sigma \in S_{n-2}} (F^{a_{\sigma_2}} \cdots F^{a_{\sigma_{n-1}}})_{a_1 a_n} A_n^{tree}(1, \sigma(2), \ldots, \sigma(n-1), n)\,, \tag{1.198}$$

which displays $(n-2)!$ sub-amplitudes, and includes the KK relation.

The trace-based and the multiperipheral-based colour decompositions are equivalent (they use the same colour-stripped amplitudes), and one can go from the trace-based to the multiperipheral-based decompositions using the KK relation. A way to see this is to use the multi-Regge kinematics (MRK), which is how the multiperipheral-based decomposition was found. Let us consider the scattering where gluons 1 and n are incoming and gluons $2, \ldots, n-1$ are outgoing. Let us divide the phase space in $(n-2)!$ simplices, according to the rapidity ordering of the $(n-2)$ outgoing gluons. Let us select the simplex with rapidity ordering $y_2 > y_3 > \ldots > y_{n-1}$. The sub-simplex with strong rapidity ordering $y_2 >> y_3 >> \ldots >> y_{n-1}$ defines the MRK.

In MRK, amplitudes have naturally a multiperipheral structure (due to Fadin-Kuraev-Lipatov [19])

$$M_n^{tree}(1_g, \ldots, n_g)|_{y_2 >> \ldots >> y_{n-1}} = g^{n-2}(F^{a_2} \cdots F^{a_{n-1}})_{a_1 a_n} A_n^{tree}(1, 2, \ldots, n-1, n)\,. \tag{1.199}$$

Note that there is only one string of F matrices.

Figure 1.17: Multiperipheral structure of n-gluon amplitude.

One can recover the multiperipheral structure from the trace-based decomposition by noting that in MRK the leading subamplitudes are of type $A_n^{tree}(1, \{a\}, n, \{\beta^T\})$, where both $\{a\}$ and $\{\beta\}$ are increasing sequences, which can be unfolded to display a KK structure.

Let us consider the colour ordering $(1, 2, \ldots, n-1, n)$ and display it in the 't Hooft graphical representation, fig. (1.14),

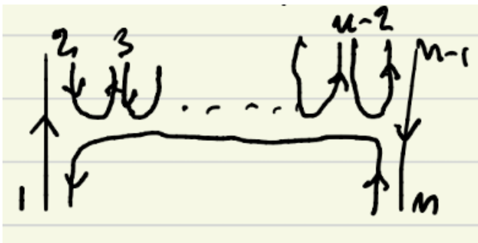


Figure 1.18: Colour ordering $(1, 2, \ldots, n-1, n)$ of n-gluon amplitude in 't Hooft graphical representation.

In MRK, the associated colour-stripped amplitude is (see eq. (H.169))

$$A_n^{tree}(1, 2, \ldots, n-1, n) = A_n^{MRK}(1, \ldots, n)\,. \tag{1.200}$$

We can think of the colour ordering $(1, 2, \ldots, n-1, n)$ as the KK relation with $\{\beta\} = \{\emptyset\}$ and $\{\alpha\} = \{2, \ldots, n-1\}$. All sets $\{\alpha\}$, where $2, \ldots, n-1$ are not in an increasing sequence, yield power sub-leading contributions because of the strong rapidity ordering, $y_2 >> y_3 >> \ldots >> y_{n-1}$.

Next, let us consider the colour ordering $(1, 2, \ldots, j-1, j+1, \ldots, n, j)$,

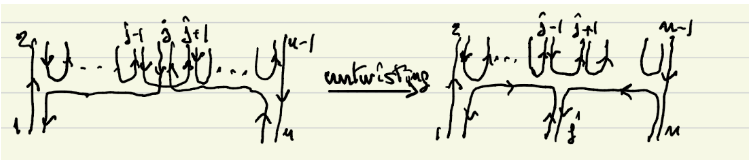


Figure 1.19: Colour ordering $(1, 2, \ldots, j-1, j+1, \ldots, n, j)$ of n-gluon amplitude in 't Hooft graphical representation.

where in the graph on the right-hand side we have "untwisted" the colour lines. In MRK, we have

$$A_n^{tree}(1, \{a\}, n, j) = -A_n^{MRK}(1, \ldots, n)\,, \qquad j = 2, \ldots, n-1\,, \tag{1.201}$$

with $\{a\} = \{2, \ldots, j-1, j+1, \ldots, n-1\}$ and $\{\beta\} = j$. We can think of $\{a\}$ ($\{\beta\}$) as the set of gluons on the upper (lower) side of the untwisted plot. The right-hand side of the equation above is the leading contribution of the KK relation,

$$A_n(1, \{a\}, n, j) = -\sum_{\sigma \in \{a\} \sqcup\!\sqcup j} A_n(1, \sigma(\{a\}, j), n)\,. \tag{1.202}$$

There are $(n-2)$ of such leading contributions since $j = 2, \ldots, n-1$, and we can suffle the position of j on the untwisted plot, and still get the result above. However, all sets $\{\alpha\}$, where $2, \ldots, j-1, j+1, \ldots, n-1$ are not in an increasing sequence, yield power sub-leading contributions.

Next, let us consider the colour ordering $(1, 2, \ldots, j-1, j+1, \ldots, k-1, k+1, \ldots, n, k, j)$,

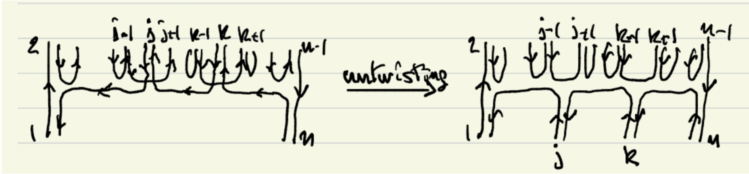


Figure 1.20: Colour ordering $(1, 2, \ldots, j-1, j+1, \ldots, k-1, k+1, \ldots, n, k, j)$ of n-gluon amplitude in 't Hooft graphical representation.

In MRK, we obtain

$$A_n^{tree}(1, \{a\}, n, \{\beta^T\}) = A_n^{MRK}(1, \ldots, n,), \qquad j, k = 2, \ldots, n-1, \qquad j < k. \tag{1.203}$$

with $\{a\} = \{2, \ldots, j-1, j+1, \ldots, k-1, k+1, \ldots, n-1\}$ and $\{\beta\} = \{j, k\}$. The right-hand side is the leading contribution of the KK relation,

$$A_n(1, \{a\}, n, \{\beta^T\}) = \sum_{\sigma \in \{a\} \sqcup\!\sqcup \{\beta\}} A_n\left(1, \sigma(\{a\}, \{\beta\}), n\right). \tag{1.204}$$

There are $\begin{pmatrix} n-2 \\ 2 \end{pmatrix}$ of such leading contributions since we can shuffle the position of j and k on the untwisted plot, and still get the result above. All sets $\{\alpha\}$, where $2, \ldots, j-1, j+1, \ldots, k-1, k+1, \ldots, n-1$ are not in an increasing sequence, yield power sub-leading contributions. By keeping moving gluons to the lower side of the untwisted plot we span all the colour orderings.

Inserting the colour factors, the amplitude in MRK is

$$\begin{aligned} M_n^{tree}(1, \ldots, n)|_{y_2 >> \ldots >> y_{n-1}} &= g^{n-2} A_n^{MRK}(1, \ldots, n) \\ &\times \Big[Tr(T^{a_1} \cdots T^{a_n}) - \sum_{j=2}^{n-1} Tr(T^{a_1} \cdots T^{a_n} T^{a_j}) + \sum_{i<k=2}^{n-1} Tr(T^{a_1} \cdots T^{a_n} T^{a_k} T^{a_j}) + \ldots\Big] \end{aligned} \tag{1.205}$$

The $\sum_{n_\beta=0}^{n-2} \begin{pmatrix} n-2 \\ n_\beta \end{pmatrix} = 2^{n-2}$ traces with alternating signs can all be collected through the identity,

$$\begin{aligned} F^{a_1 a_2 x_1} F^{x_1 a_3 x_2} \cdots F^{x_{n-3} a_{n-1} a_n} &= Tr\Big(T^{a_1}\big[T^{a_2}, [T^{a_3}, [\ldots, [T^{a_{n-1}}, T^{a_n}], \ldots]\Big) \\ &= (F^{a_2} \cdots F^{a_{n-1}})_{a_1 a_n}, \end{aligned} \tag{1.206}$$

thus we get

$$M_n^{tree}(1, \ldots, n)|_{y_2 >> \ldots >> y_{n-1}} = g^{n-2} (F^{a_2} \cdots F^{a_{n-1}})_{a_1 a_n} A_n^{MRK}(1, \ldots, n). \tag{1.207}$$

The same procedure can be repeated for all the $(n-2)!$ simplices, thus generating the $(n-2)!$ strings of F's of the multiperipheral-based colour decomposition (1.198).

By using the identity (1.206), it is possible to show that the multiperipheral-based colour decomposition is equivalent to inserting the KK relation into the trace-based colour decomposition also without resorting to a specific kinematic set-up. More details can be found in section 2.3 of [12] and in [18].

Finally, the KK relation has been proven using monodromy relations (contour deformations in the complex plane, i.e. on the Riemann sphere with punctures) in the low-energy limit of string-theory amplitudes ([20, 21]) and using the BCFW recursion relations in field theory [22].

This is not the end of the story. There is actually another set of relations, induced by colour-kinematics (CK) duality, which reduces the number of independent sub-amplitudes to $(n-3)!$. They are a bit more involved, and there is no known colour decomposition which allows us to write the amplitude in terms of $(n-3)!$ sub-amplitudes. However, CK duality holds great value, because it exposes a link between gauge theories and gravity. We will postpone the study of CK duality until we have got acquainted with the pure-gluon sub-amplitudes.

1.7.3 Parke-Taylor formula

Gluon amplitudes are built out of 3-gluon and 4-gluon vertices. n-gluon tree amplitudes may have up to $n-2$ vertices. Each vertex contributes at most one momentum, so there can be at most $(n-2)$ momenta to contract with the n polarisation vectors $\epsilon(p_i, k_i)$, $i = 1, \ldots, n$, so each diagram will contain at least one $\epsilon_i \cdot \epsilon_j$ term. If we can arrange that all the $\epsilon_i \cdot \epsilon_j$ vanish, then the amplitude vanishes.

Let us consider the tree all-plus helicity amplitude. Using the identity (1.116), $\epsilon^+(p_i, k)\cdot\epsilon^+(p_j, k) = 0$, and choosing the same reference vector $k_i = k$ for all ϵ's, we can make the amplitude vanish, $A_n^{tree}(1^+, 2^+, \ldots, n^+) = 0$ (of course, we cannot choose k to be any of the momenta p_i, else the polarisation $\epsilon(p_i, p_i)$ would be singular, however we can build a light-like vector k^μ as a linear combination of the momenta p_i). Next, let us take gluon 1 to have negative helicity, i.e. we consider the sub-amplitude $A_n^{tree}(1^-, 2^+, \ldots, n^+)$. We may choose e.g. the reference vectors to be $k_1 = p_n$ and $k_i = p_1$ with $i = 2, \ldots, n$. Using the identity (1.117), $\epsilon^+(p_i, p_1) \cdot \epsilon^-(p_1, p_n) = 0$, we make also this amplitude vanish $A_n^{tree}(1^-, 2^+, \ldots, n^+) = 0$.

The sub-amplitude with the two positive and two-negative helicity gluons is (see app. H.14)

$$A_4^{tree}(1, 2, 3, 4) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{1.208}$$

where i, j are the negative-helicity gluons independently of their position in the colour ordering. The amplitude above is the beginning of an infinite tower of MHV amplitudes, given by the renowned Parke-Taylor (PT) formula [1] for the sub-amplitude with two-negative and $(n-2)$ positive helicity gluons,

$$A_4^{tree}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle (n-1)n\rangle\langle n1\rangle}\,. \tag{1.209}$$

Likewise, for the $\overline{\text{MHV}}$ sub-amplitude with two positive and $(n-2)$-negative helicity gluons,

$$A_4^{tree}(1^-, 2^-, \dots, i^+, \dots, j^+, \dots, n^-) = (-1)^n \frac{[ij]^4}{[12][23]\dots[(n-1)n][n1]} \,. \qquad (1.210)$$

As we already said in sec. 1.6.3 when discussing $q\bar{q} \to (n-2)$ gluons, for MHV amplitudes there are no multi-particle poles, $s_{12,\dots,m} = (k_1 + \dots + k_m)^2$, with $m \geq 3$.

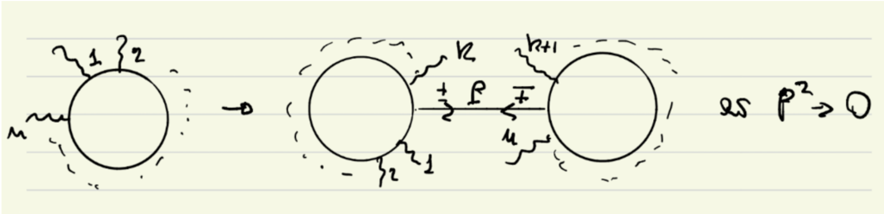


Figure 1.21: Multi-particle factorisation. The on-shell leg P is depicted as outgoing on both sides, but then with opposite helicities.

A MHV amplitude has two-negative helicity gluons, plus one more negative helicity from the on-shell leg P. That is three negative helicity gluons, but in order for the MHV amplitude to factorise into two non-vanishing amplitudes, each must have at least two-negative helicity gluons, for a total of at least four. Thus, MHV amplitudes cannot have multi-particle poles.

Two questions arise:

1. The PT formula is extremely simple. It looks magic. Can we prove it? Shortly after it was found, the PT formula was proven by Berends and Giele [23] using recursion relations on off-shell currents. Although they lie slightly outside our course (which deal with on-shell quantities), we shall display later the Berends-Giele recursion relations, since they are still the fastest way to generate tree amplitudes. We shall look at Berends-Giele proof of the PT formula later.

2. There are formulae for MHV amplitudes of gluons only; of $q\bar{q} \to (n-2)$ gluons; of four fermions plus any number of gluons or photons, and they look alike. Are they related ? Yes, they are, and one can see it is using the relations imposed by supersymmetric Yang-Mills theories. Now, QCD is a non-supersymmetric Yang-Mills theory, but one can see that only at loop level, where one can appreciate the difference between a quark loop and a gluino loop. So n-gluon tree amplitudes cannot tell if they come from QCD, or from a supersymmetric extension. Tree amplitudes with quarks can tell the difference, because quarks are in the fundamental representation of $SU(N_c)$, while gluinos are in the adjoint; but only after including colour. Thus, colour-stripped amplitudes with quarks are indistinguishable from colour-stripped amplitudes with gluiuos. We shall look at these properties in the next section.

1.7.4 Bern, Carrasco, Johansson relations

Let us go back to the issue of how many independent colour-stripped amplitudes there are in a scattering amplitude with only gluons. Let us consider the four-gluon amplitude. Four diagrams contribute to it,

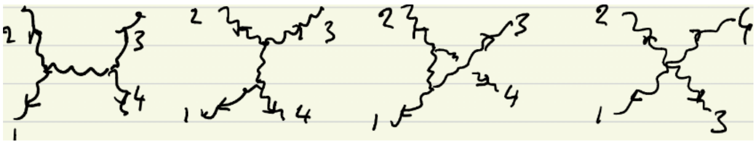


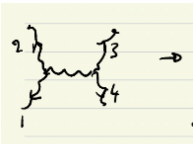
Figure 1.22: Feynman diagrams contributing to the four-gluon amplitude.

We can write the amplitude as a sum over the three channels corresponding to the first three diagrams,

$$iM_4 = -ig^2\Big(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\Big)\,, \tag{1.211}$$

where c_s, c_t, c_u (n_s, n_t, n_u) are colour (kinematic) factors.

The diagram with the four-gluon vertex is absorbed into the diagrams with three-gluon vertices by matching the suitable colour factor and inserting a propagator as s_{ij}/s_{ij}. For example, using the usual three-gluon vertex (1.160), the s-channel contribution is ($s_{12} = s$)



$$\begin{aligned} & g^2 f^{a_1 a_2 b} f^{b a_3 a_4} \frac{-i}{s} \epsilon_1^{\mu_1} \epsilon_2^{\mu_2} \epsilon_3^{\mu_3} \epsilon_4^{\mu_4} \\ & \cdot[(p_1 - p_2)_a g^{\mu_1\mu_2} + 2p_2^{\mu_2} g^{a\mu_2} - 2p_1^{\mu_2} g^{a\mu_1}] \\ & \cdot[(p_3 - p_4)_a g^{\mu_3\mu_4} + 2p_4^{\mu_3} g^{a\mu_4} - 2p_3^{\mu_4} g^{a\mu_3}]\,. \end{aligned} \tag{1.212}$$

where we used that the gluons are on-shell, $p_i \cdot \epsilon_i = 0$. Then, out of the 4-gluon vertex (1.165), we just pick up the matching colour factor,

$$-ig^2 f^{a_1 a_2 b} f^{b a_3 a_4} (g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3})\,. \tag{1.213}$$

If the s-channel colour factor is taken to be

$$c_s = F^{a_1 a_2 b} F^{b a_3 a_4} = -2 f^{a_1 a_2 b} f^{b a_3 a_4}\,, \tag{1.214}$$

with the normalisation (1.197), the s-channel kinematic factor is

$$\begin{aligned} n_s \;=\; & -\frac{1}{2}\Big([(p_1 - p_2)^\alpha \epsilon_1 \cdot \epsilon_2 + 2\epsilon_1 \cdot p_2 \epsilon_2^\alpha - 2\epsilon_2 \cdot p_1 \epsilon_1^\alpha]\,[(p_3 - p_4)_\alpha \epsilon_3 \cdot \epsilon_4 + 2\epsilon_3 \cdot p_4 \epsilon_{4\alpha} - 2\epsilon_4 \cdot p_3 \epsilon_{3\alpha}] \\ & + s(\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3)\Big)\,. \end{aligned} \tag{1.215}$$

The other colour and kinematic factors are obtained by a cyclic permutation of the labels $(1, 2, 3)$,

$$c_t n_t = c_s n_s|_{1\to2\to3\to1}\,, \qquad c_u n_u = c_s n_s|_{1\to3\to2\to1}\,. \tag{1.216}$$

The QCD Ward identities say that if we replace the polarisation of a gluon, say ϵ_4, by its momentum p_4, while keeping all the other gluons on-shell, the amplitude vanishes.

If we replace ϵ_4 with p_4 in n_s, we get

$$\begin{aligned} n_s|_{\epsilon_4=p_4} &= -\frac{1}{2}\Big([(p_1-p_2)^\alpha\epsilon_1\cdot\epsilon_2+2\epsilon_1\cdot p_2\epsilon_2^\alpha-2\epsilon_2\cdot p_1\epsilon_1^\alpha]\,[(p_3+p_4)_\alpha\epsilon_3\cdot p_4-2p_3\cdot p_4\epsilon_{3\alpha}] \\ &\quad +s(\epsilon_1\cdot\epsilon_3\epsilon_2\cdot p_4-\epsilon_1\cdot p_4\epsilon_2\cdot\epsilon_3)\Big)\,. \end{aligned} \tag{1.217}$$

Using momentum conservation $p_4=-p_1-p_2-p_3$,

$$\begin{aligned} n_s|_{\epsilon_4=p_4} &= -\frac{1}{2}\Big([\cancelto{0}{-(p_1-p_2)\cdot(p_1+p_2)}\epsilon_3\cdot p_4-s\epsilon_3\cdot(p_1-p_2)]\epsilon_1\cdot\epsilon_2 \\ &\quad +(-2\cancel{\epsilon_1\cdot p_2p_1\cdot\epsilon_2}+2\cancel{\epsilon_2\cdot p_1\epsilon_1\cdot p_2})\epsilon_3\cdot p_4-s\left(2\epsilon_1\cdot p_2\epsilon_2\cdot\epsilon_3-2\epsilon_2\cdot p_1\epsilon_1\cdot\epsilon_3\right) \\ &\quad +s\left(-\epsilon_1\cdot\epsilon_3\epsilon_2\cdot(p_1+p_3)+\epsilon_2\cdot\epsilon_3\epsilon_1\cdot(p_2+p_3)\right)\Big) \\ &= -\frac{s}{2}\Big(\epsilon_3\cdot(p_2-p_1)\epsilon_1\cdot\epsilon_2+\epsilon_1\cdot(p_3-p_2)\epsilon_2\cdot\epsilon_3+\epsilon_2\cdot(p_1-p_3)\epsilon_3\cdot\epsilon_1\Big)\,, \end{aligned} \tag{1.218}$$

which can be written as

$$n_s|_{\epsilon_4=p_4}=sf(p,\epsilon)\,, \tag{1.219}$$

with

$$f(p,\epsilon)=-\frac{1}{2}\sum_{\text{cyclic }\ \sigma}\epsilon_{\sigma_1}\cdot\epsilon_{\sigma_2}(\epsilon_{\sigma_3}\cdot p_{\sigma_2}-\epsilon_{\sigma_3}\cdot p_{\sigma_1})\,, \tag{1.220}$$

which is by definition invariant under cyclic permutations. Thus, for the amplitude, we get

$$\Big(\frac{n_sc_s}{s}+\frac{n_tc_t}{t}+\frac{n_uc_u}{u}\Big)|_{\epsilon_4=p_4}=(c_s+c_t+c_u)f(p,\epsilon)\,, \tag{1.221}$$

which is gauge invariant if $c_s+c_t+c_u=0$. But

$$c_s+c_t+c_u=-2(f^{a_1a_2b}f^{ba_3a_4}+f^{a_2a_3b}f^{ba_1a_4}+f^{a_3a_1b}f^{ba_2a_4})=0\,, \tag{1.222}$$

because of Jacobi identity on the structure functions.

Further, also the sum of the kinematic factors vanishes,

$$n_s+n_t+n_u=0\,. \tag{1.223}$$

E.g. collecting the terms proportional to $\epsilon_1\cdot\epsilon_2$, from n_s we get

$$(p_1-p_2)\cdot(p_3-p_4)\epsilon_3\cdot\epsilon_4+2(p_1-p_2)\cdot\epsilon_4\epsilon_3\cdot p_4-2(p_1-p_2)\cdot\epsilon_3\epsilon_4\cdot p_3\,, \tag{1.224}$$

from n_t, we get

$$4\epsilon_3\cdot p_2\epsilon_4\cdot p_1+2p_2\cdot p_3\epsilon_3\cdot\epsilon_4\,, \tag{1.225}$$

from n_u, we get

$$-4\epsilon_3\cdot p_1\epsilon_4\cdot p_2-2p_3\cdot p_1\epsilon_3\cdot\epsilon_4\,. \tag{1.226}$$

So, using momentum conservation, the term of $n_s + n_t + n_u$ proportional to $\epsilon_1 \cdot \epsilon_2$ is

$$
\begin{aligned}
&\big[2(\cancel{p_1} - \cancel{p_2}) \cdot p_3 \epsilon_3 \cdot \epsilon_4 - 2(p_1 - p_2) \cdot \epsilon_4 \epsilon_3 \cdot (p_1 + p_2) - 2(p_1 - p_2) \cdot \epsilon_3 \epsilon_4 \cdot p_3 \\
&\quad + 4\epsilon_3 \cdot p_2 \epsilon_4 \cdot p_1 + \cancel{2p_2 \cdot p_3 \epsilon_3 \cdot \epsilon_4} - 4\epsilon_3 \cdot p_1 \epsilon_4 \cdot p_2 - \cancel{2p_3 \cdot p_1 \epsilon_3 \cdot \epsilon_4}\big] \epsilon_1 \cdot \epsilon_2 \\
&= \big([-2(p_1 - p_2) \cdot \epsilon_4 + 2p_3 \cdot \epsilon_4 + 4p_1 \cdot \epsilon_4] p_2 \cdot \epsilon_3 + [-2(p_1 - p_2) \cdot \epsilon_4 - 2p_3 \cdot \epsilon_4 - 4p_2 \cdot \epsilon_4] p_1 \cdot \epsilon_3\big) \epsilon_1 \cdot \epsilon_2 \\
&= 2(p_1 + p_2 + p_3) \cdot \epsilon_4 (p_2 \cdot \epsilon_3 - p_1 \cdot \epsilon_3) \epsilon_1 \cdot \epsilon_2 \\
&= 0\,. \qquad (1.227)
\end{aligned}
$$

Likewise, all the other terms vanish (please check it).

Solving the Jacobi identity $c_t = -c_u - c_s$, we can write the amplitude in a gauge-invariant form since the colour factors c_u and c_s are independent,

$$
iM_4 = -ig^2 \left[\left(\frac{n_s}{s} - \frac{n_t}{t} \right) c_s + \left(\frac{n_u}{u} - \frac{n_t}{t} \right) c_u \right] \,, \qquad (1.228)
$$

i.e. since the amplitude is gauge invariant, and c_s and c_u are independent, then the coefficients $\left(\frac{n_s}{s} - \frac{n_t}{t} \right)$ and $\left(\frac{n_u}{u} - \frac{n_t}{t} \right)$ are gauge invariant.

But the multiperipheral colour decomposition (1.198) for the four-point amplitude is

$$
M_4 = g^2 \big[F^{a_1 a_2 b} F^{b a_3 a_4} A_4(1,2,3,4) + F^{a_1 a_3 b} F^{b a_2 a_4} A_4(1,3,2,4) \big] \,, \qquad (1.229)
$$

Since $c_s = F^{a_1 a_2 b} F^{b a_3 a_4}$ and $c_u = F^{a_3 a_1 b} F^{b a_2 a_4}$, we can also write it as

$$
M_4 = g^2 [c_s A_4(1,2,3,4) - c_u A_4(1,3,2,4)] \,. \qquad (1.230)
$$

Equating eqs. (1.228) and (1.230), we get the colour-stripped amplitudes,

$$
iA_4(1,2,3,4) = -i \left(\frac{n_s}{s} - \frac{n_t}{t} \right) = -i \left[\left(\frac{1}{s} + \frac{1}{t} \right) n_s + \frac{n_u}{t} \right] \,, \qquad (1.231)
$$

$$
iA_4(1,3,2,4) = i \left(\frac{n_u}{u} - \frac{n_t}{t} \right) = i \left[\frac{n_s}{t} + \left(\frac{1}{u} + \frac{1}{t} \right) n_u \right] \,, \qquad (1.232)
$$

where we used the kinematic identity (1.223) $n_t = -n_u - n_s$. We can also write them in matrix form,

$$
i \begin{pmatrix} A_4(1,2,3,4) \\ A_4(1,3,2,4) \end{pmatrix} = -i \begin{pmatrix} -\frac{u}{st} & \frac{1}{t} \\ -\frac{1}{t} & \frac{s}{ut} \end{pmatrix} \begin{pmatrix} n_s \\ n_u \end{pmatrix} \,, \qquad (1.233)
$$

where we used momentum conservation. Note that the rank of the matrix is less than 2, since the determinant vanishes,

$$
-\frac{u}{st} \frac{s}{ut} + \frac{1}{t^2} = 0 \,, \qquad (1.234)
$$

consistently with the fact that the system (1.233) cannot be solved for n_s and n_u, which are gauge dependent, in terms of the sub-amplitudes, which are gauge invariant. The two equations in (1.233) can be put together,

$$A_4(1,3,2,4) = \frac{s}{u} A_4(1,2,3,4)\,, \tag{1.235}$$

or equivalently, the eigenvalue equation for the matrix in (1.233),

$$\lambda^2 - \left(\frac{s}{ut} - \frac{u}{st}\right)\lambda - \cancelto{0}{\frac{u}{st}\frac{s}{ut} + \frac{1}{t^2}} = 0\,, \tag{1.236}$$

has solutions,

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{1}{s} - \frac{1}{u}\,. \tag{1.237}$$

The orthogonality condition on the null eigenvector,

$$-\frac{u}{st}v_1 + \frac{1}{t}v_2 = 0\,, \tag{1.238}$$

yields eq. (1.235).

Eq. (1.235) was found by Bern, Carrasco, Johansson [28], it is known as BCJ relation and implies a linear dependence between the two sub-amplitudes of the four-gluon amplitude.

Firstly, let us check that the BCJ relation (1.235) we obtained out of the four-gluon amplitude does hold on a specific helicity configuration. Let us take $(1^-2^-3^+4^+)$. The sub-amplitudes are simultaneously MHV and $\overline{\text{MHV}}$, and can be written as

$$A_4(1^-2^-3^+4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{1.239}$$

$$\begin{aligned} A_4(1^-3^+2^-4^+) &= \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle} \\ &= -\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle} A_4(1^-2^-3^+4^+) \\ &= \frac{[34]\langle 34\rangle}{[24]\langle 24\rangle} A_4(1^-2^-3^+4^+) \\ &= \frac{s}{u} A_4(1^-2^-3^+4^+)\,, \end{aligned} \tag{1.240}$$

where in the second line we used momentum conservation, $\langle 12\rangle[24] = -\langle 13\rangle[34]$.

Secondly, the BCJ relation generalises to n-gluon amplitudes,

$$\sum_{i=3}^{n}\Big(\sum_{j=3}^{i} s_{2j}\Big) A_n(1,3,\ldots,i,2,i+1,\ldots,n) = 0\,, \tag{1.241}$$

or its equivalent form,

$$\sum_{i=2}^{n-1}\Big(\sum_{j=2}^{i} s_{1j}\Big)A_n(2,\ldots,i,1,i+1,\ldots,n)=0\,, \tag{1.242}$$

plus permutations, yielding $(n-2)!$ relations, which are called fundamental BCJ relations.

For $n=4$, we get

$$s_{23}A(1,3,2,4)+\overbrace{(s_{23}+s_{24})}^{-s_{12}}A(1,3,4,2)=0\,, \tag{1.243}$$

then we use the photon decoupling identity,

$$A(1,2,3,4)+A(1,3,2,4)+A(1,3,4,2)=0 \tag{1.244}$$

and we get

$$s_{12}A(1,2,3,4)+\overbrace{(s_{12}+s_{23})}^{-s_{13}}A(1,3,2,4)=0 \tag{1.245}$$

i.e. eq. (1.235).

For $n=5$, we get

$$s_{23}A(1,3,2,4,5)+(s_{23}+s_{24})A(1,3,4,2,5)+\overbrace{s_{23}+s_{24}+s_{25}}^{-s_{12}}A(1,3,4,5,2)=0\,. \tag{1.246}$$

Then, we use the KK relation (or photon decoupling)

$$A(1,3,4,5,2)=-A(1,3,4,2,5)-A(1,3,2,4,5)-A(1,2,3,4,5)\,, \tag{1.247}$$

plus five more relations through the permutations of the indices 3, 4, 5. Out of the six relations, we get four sub-amplitudes in terms of two independent ones (from [28])

$$\begin{aligned}
A_5^{tree}(1,3,4,2,5) &= \frac{-s_{12}s_{45}A_5^{tree}(1,2,3,4,5)+s_{14}(s_{24}+s_{25})A_5(1,4,3,2,5)}{s_{13}s_{24}}\,,\\
A_5^{tree}(1,2,4,3,5) &= \frac{-s_{14}s_{25}A_5^{tree}(1,4,3,2,5)+s_{45}(s_{12}+s_{24})A_5(1,2,3,4,5)}{s_{24}s_{35}}\,,\\
A_5^{tree}(1,4,2,3,5) &= \frac{-s_{12}s_{45}A_5^{tree}(1,2,3,4,5)+s_{25}(s_{14}+s_{24})A_5(1,4,3,2,5)}{s_{35}s_{24}}\,,\\
A_5^{tree}(1,3,2,4,5) &= \frac{-s_{14}s_{25}A_5^{tree}(1,4,3,2,5)+s_{12}(s_{24}+s_{45})A_5(1,2,3,4,5)}{s_{13}s_{24}}\,.
\end{aligned} \tag{1.248}$$

There is a more general form of the BCJ relation, but it is possible to show that the fundamental BCJ relations suffice to generate the linear relations among sub-amplitudes which reduce the independent sub-amplitudes to $(n-3)!$. In fact, the $(n-2)!$ relations yield a $(n-2)!\times(n-2)!$ matrix having rank $(n-3)!$, which means that there are $(n-2)!-(n-3)!=(n-3)\;(n-3)!$ relations among the sub-amplitudes, which reduce them to $(n-3)!$ independent ones.

The fundamental BCJ relations (as well as the KK relations, the photon decoupling identity and the reflection identity) can be proven through the BCFW recursion relations (see [22]).

1.8 Supersymmetric relations

Tree amplitudes cannot tell if they come from QCD, or from a supersymmetric extension. That is true on the whole amplitude for gluons only, and on the colour-stripped amplitude for quarks with gluons. Thus, in order to study the gluon amplitudes and the connections between different MHV amplitudes, we can use any supersymmetric extension of Yang-Mills we like. We shall use $N = 1$ supersymmetry, which is the simplest.

Let us consider a theory with a local SU(N_c) symmetry and a global N=1 supersymmetry, which connects bosons and fermions. The N=1 supersymmetry is characterised by a supercharge Q_a, where a is a right-handed spinor index, and its hermitian conjugate $Q^\dagger_{\dot{a}}$.

Q_a and $Q^\dagger_{\dot{a}}$, together with the generators of the Poincare group P^μ and $M^{\mu\nu}$, form the supersymmetry algebra,

$$[Q_a, P^\mu] = 0\,, \tag{1.249}$$
$$[Q^\dagger_{\dot{a}}, P^\mu] = 0\,, \tag{1.250}$$
$$[Q_a, M^{\mu\nu}] = (S^{\mu\nu}_R)_a^c Q_c\,, \tag{1.251}$$
$$[Q^\dagger_{\dot{a}}, M^{\mu\nu}] = (S^{\mu\nu}_L)_{\dot{a}}^{\dot{c}} Q^\dagger_{\dot{c}}\,, \tag{1.252}$$
$$\{Q_a, Q_b\} = 0\,, \tag{1.253}$$
$$\{Q_a, Q^\dagger_{\dot{a}}\} = 2\sigma^\mu_{a\dot{a}} P_\mu\,, \tag{1.254}$$

where eqs. (1.249) and (1.250) say that the supercharges are conserved, and eqs. (1.251) and (1.252) that they transform indeed as spinors under Lorentz transformations.

Then we enlarge the space x^μ in order to include anticommuting (or Grassmann) right-handed spinor coordinates η_a, and the left-handed complex conjugate $\eta^*_{\dot{a}}$. $\{x, \eta, \eta^*\}$ form the superspace. We require that $Q(\eta) = \overline{\eta}^a Q_a$, so that $Q(\eta)$ commutes with both bosonic and fermionic fields.

The supercharge $Q(\eta)$ connects the gluon $g^\pm$ to a massless Weyl spinor $\Lambda^\pm$, the gluino,

$$[Q(\eta), g^\pm(p)] = \mp\Gamma^\pm(p,\eta)\Lambda^\pm(p)\,, \tag{1.255}$$
$$[Q(\eta), \Lambda^\pm(p)] = \mp\Gamma^\mp(p,\eta)g^\pm(p)\,. \tag{1.256}$$

where $\Gamma^\pm(p,\eta)$ is linear in η.

Through a Jacobi identity for the supersymmetry algebra (see Dixon's TASI 1995 lectures [8]), $\Gamma^\pm(p,\eta)$ can be chosen to be

$$\Gamma^+(p,\eta) = \overline{\eta} u_-(p)\,, \qquad \Gamma^-(p,\eta) = \overline{u_-}(p)\eta\,. \tag{1.257}$$

The parameter η is arbitrary: we may choose it to be $\overline{\eta} = \theta u^\dagger_+(k) = \theta\langle k^+|$ and $\eta = \theta u_+(k) = \theta\,|k^+\rangle$ i.e. a spinor with an arbitrary light-like vector k, times a Grassmann variable θ, so that $\Gamma^\mp(p,q)$

(anti)-commutes with (fermionic) bosonic operators,

$$\Gamma^+(p,\eta(k)) = \theta\overline{u}_+(k)u_-(p) = \theta[kp]\,, \tag{1.258}$$

$$\Gamma^-(p,\eta(k)) = \theta\langle pk\rangle\,. \tag{1.259}$$

The supercharge $Q(\eta(k))$ annihilates the vacuum, so the commutator of Q with any string of operators ehich create or annihilate $g^\pm$ or $\Lambda^\pm$ has a vanishing vacuum expectation value (vev),

$$\langle 0|\,[Q,\phi_1,\ldots,\phi_n]\,|0\rangle = \sum_{i=1}^{n}\langle 0|\,\phi_1,\ldots,[Q,\phi_i]\ldots\phi_n\,|0\rangle = 0\,. \tag{1.260}$$

where $\phi_i = g^\pm, \Lambda^\pm$. This is a supersymmetric Ward identity (SWI).

Let us apply it with a string of operators as follows:

$$\begin{aligned} 0 &= \langle 0|\,[Q,\Lambda_1^+ g_2^+\ldots g_n^+]\,|0\rangle \\ &= -\Gamma^-(p_1,k)A_n(g_1^+,g_2^+,\ldots,g_n^+) + \langle 0|\sum_{i=2}^{n}\Lambda_1^+ g_2^+\ldots g_{i-1}^+(-\Gamma^+(p_i,k))\Lambda_i^+ g_{i+1}^+\ldots g_n^+\,|0\rangle \\ &= -\Gamma^-(p_1,k)A_n(g_1^+,g_2^+,\ldots,g_n^+) + \sum_{i=2}^{n}\Gamma^+(p_i,k)\underbrace{A_n(\Lambda_1^+ g_2^+\ldots g_{i-1}^+\Lambda_i^+ g_{i+1}^+\ldots g_n^+)}_{\to 0}\,, \end{aligned} \tag{1.261}$$

where we used that $\{\Lambda_1^+,\Gamma^+(p_i,k)\} = 0$, and that on the gluino (fermion) line, helicity is conserved and thus the state $\Lambda_1^+\Lambda_i^+$ is forbidden. So the SWI implies that

$$A_n(g_1^+,g_2^+,\ldots,g_n^+) = 0\,. \tag{1.262}$$

We did not specify a loop expansion, so this is true to all loops for a super-Yang-Mills (SYM) theory, and at tree level for QCD, in agreement (for QCD) with the result we had derived in sec. 1.7.3.

Next, let us take the string of operators,

$$\langle 0|\,[Q,\Lambda_1^+ g_2^- g_3^+\ldots g_n^+]\,|0\rangle = 0 \tag{1.263}$$

using the commutators of the supercharge Q, we get

$$\begin{aligned} = &-\Gamma^-(p_1,k)A_n(g_1^+,g_2^-,g_3^+,\ldots,g_n^+) \\ &-\Gamma^-(p_2,k)A_n(\Lambda_1^+,\Lambda_2^-,g_3^+,\ldots,g_n^+) \\ &+\sum_{i=3}^{n}\Gamma^+(p_i,k)\underbrace{A_n(\Lambda_1^+,g_2^-,g_3^+,\ldots,g_{i-1}^+,\Lambda_i^+,g_{i+1}^+,\ldots,g_n^+)}_{\to 0}\,, \end{aligned} \tag{1.264}$$

then if we choose $k = p_2 \to \Gamma^-(p_2,k) = 0$, so

$$A_n(g_1^+,g_2^-,g_3^+,\ldots,g_n^+) = 0\,, \tag{1.265}$$

which is true to all loops for SYM, and at tree level for QCD, again in agreement with the result we had guessed in sec. 1.7.3.

If we choose $k = p_1 \to \Gamma^-(p_1, k) = 0$, so

$$A_n(\Lambda_1^+, \Lambda_2^-, g_3^+, \ldots, g_n^+) = 0\,, \tag{1.266}$$

which is true to all loops for SYM, and which at tree level for QCD, it implies that

$$A_n^{tree}(q_1^+, \overline{q}_2^-, g_3^+, \ldots, g_n^+) = 0\,, \tag{1.267}$$

again in agreement with the result we had guessed in sec. 1.6.3.

Let us take the string of operators,

$$\begin{aligned} 0 &= \langle 0|\,[Q, g_1^- g_2^- \Lambda_3^+ g_4^+ \ldots g_n^+]\,|0\rangle \\ &= \Gamma^-(p_1,k) A_n(\Lambda_1^-, g_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+) \\ &+ \Gamma^-(p_2,k) A_n(g_1^-, \Lambda_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+) \\ &- \Gamma^-(p_3,k) A_n(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_n^+)\,. \end{aligned} \tag{1.268}$$

If we choose $k = p_1 \to \Gamma^-(p_1, k) = 0$ and we get

$$\langle 21\rangle A_n(g_1^-, \Lambda_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+) - \langle 31\rangle A_n(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_n^+) = 0\,. \tag{1.269}$$

i.e.

$$A_n(g_1^-, \Lambda_2^-, \Lambda_3^+, g_4^+, \ldots, g_n^+) = \frac{\langle 13\rangle}{\langle 12\rangle} A_n(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_n^+)\,, \tag{1.270}$$

which is true to all loops for SYM, and which at tree level for QCD implies that

$$\begin{aligned} A_n^{tree}(g_1^-, \overline{q}_2^-, q_3^+, g_4^+, \ldots, g_n^+) &= \frac{\langle 13\rangle}{\langle 12\rangle} A_n(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_n^+) \\ &= \frac{\langle 12\rangle^3 \langle 13\rangle}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}\,, \end{aligned} \tag{1.271}$$

i.e. the result we had stated, without proof, previously.

Through the SWI, we have managed to link the n-gluon MHV amplitudes to the MHV amplitudes for $q\bar{q} \to (n-2)$ gluons. So it will be enough to prove one of them, say the n-gluon MHV amplitude.

More details on N=1 supersymmetry can be found e.g. in [4], and its application to helicity amplitudes in [7] and in [8], [9].

1.9 Berends-Giele recursion relations

Let us consider the sum $J^\mu(1,2,\ldots,n)$ of colour-ordered $(n+1)$-point Feynman diagrams, where legs $1,2,\ldots,n$ are on-shell and one leg is off-shell, with uncontracted vector index μ. Since J^μ is an off-shell quantity it is gauge dependent; in particular, it depends on the reference vectors of the on-shell gluons, until we extract an on-shell quantity.

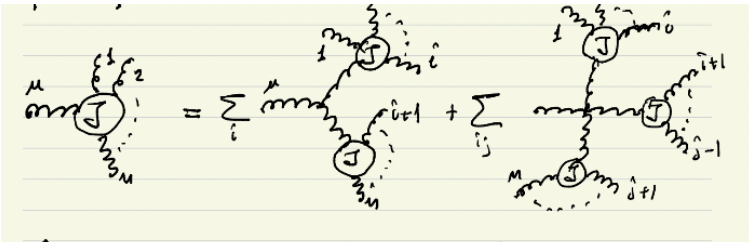


Figure 1.23: Off-shell current in Berends-Giele recursion relations.

The current J^μ can be constructed recursively, following the off-shell line back into the diagram. Leg μ must be attached to either a cubic or a quartic vertex, whose legs are then contracted with similar currents with fewer legs,

$$\begin{aligned} J^\mu = \quad & -\frac{i}{P_{1,n}^2}\left[\sum_{i=1}^{n-1} V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})J_\nu(1,\ldots,i)J_\rho(i+1,\ldots,n)\right. \\ & \left. +\sum_{i=1}^{n-2} V_4^{\mu\nu\rho\sigma} J_\nu(1,\ldots,i)\sum_{j=i+1}^{n-1} J_\rho(i+1,\ldots,j)J_\sigma(j+1,\ldots,n)\right], \end{aligned} \tag{1.272}$$

with $P_{i,j} = p_i + \ldots + p_j$ and the colour ordered vertices,

$$V_3^{\mu\nu\rho}(P,Q) = \frac{i}{\sqrt{2}}[2g^{\mu\rho}Q^\nu - 2g^{\mu\nu}P^\rho + g^{\nu\rho}(P-Q)^\mu], \tag{1.273}$$

$$V_4^{\mu\nu\rho\sigma} = \frac{i}{2}[2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}]. \tag{1.274}$$

where we used the 4-gluon and 3-gluon colour-ordered vertices we introduced in sec. 1.6.1, with momentum conservation $P_{1,n} = -(P+Q)$ and treating legs P and Q as on-shell.

The currents fulfil a reflection identity,

$$i)\qquad J^\mu(n,\ldots,2,1) = (-1)^{n+1}J^\mu(1,2,\ldots,n), \tag{1.275}$$

and they are conserved,

$$ii)\qquad P_{1,n}^\mu J_\mu(1,2,\ldots,n) = 0. \tag{1.276}$$

The photon decoupling identity yields a vanishing linear combination of n currents,

$$iii)\quad J^\mu(1,2,3,\ldots,n) + J^\mu(2,1,3,\ldots,n) + \ldots + J^\mu(2,3,\ldots,1,n) + J^\mu(2,3,\ldots,n,1) = 0. \tag{1.277}$$

$(n+1)$-point amplitudes are obtained from the currents $J_\mu(1,2,\ldots,n)$ by amputating the off-shell propagator, i.e. by multiplying by $iP^2_{1,n}$, then by contracting with $\epsilon^\mu(p_{n+1})$ and taking the limit $P^2_{1,n} = p^2_{n+1} \to 0$. Closed-form expressions of off-shell currents are known for the simplest helicity configurations, $J^\mu(1^+,2^+,\ldots,n^+)$ and $J^\mu(1^-,2^+,\ldots,n^+)$, as we will see. From the former, one can show (once more) that sub-amplitudes with none or one negative-helicity gluon, $A^{tree}_{n+1}(1^+,2^+,\ldots,(n+1)^\pm) = 0$. From the latter, one can prove the Parke-Taylor formula.

Let us construct the currents for the simplest examples. The current $J^\mu(1,2,\ldots,n)$ must include the n polarisation vectors $\epsilon^\mu_\pm(P_i,q_i)$, $i=1,\ldots,n$. Let us set

$$J^\mu(i^\pm) = \epsilon^\mu_\pm(p_i,q_i)\,. \tag{1.278}$$

Then

$$J^\mu(1,2) = -\frac{i}{(p_1+p_2)^2} V_3^{\mu\nu\rho}(p_1,p_2) J_\nu(1) J_\rho(2)\,. \tag{1.279}$$

For positive-helicity gluons, we take the same reference vector q. In $V_3^{\mu\nu\rho}$, the $g^{\nu\rho}$ term does not contribute, because it contracts the polarisation vectors, yielding $\epsilon_+(1,q)\cdot\epsilon_+(2,q) = 0$. In app. H.19, we show that

$$J^\mu(1^+,2^+) = \frac{1}{\sqrt{2}} \frac{\langle q^- |\, \gamma_\mu (\not{p}_1 + \not{p}_2)\, | q^+\rangle}{\langle q1\rangle\langle 12\rangle\langle 2q\rangle}\,. \tag{1.280}$$

We make the ansatz that this formula extends to n gluons,

$$J^\mu(1^+,2^+,\ldots,n^+) = \frac{1}{\sqrt{2}} \frac{\langle q^- |\, \gamma_\mu \not{P}_{1,n}\, | q^+\rangle}{\langle q1\rangle\langle 12\rangle\cdots\langle (n-1),n\rangle\langle nq\rangle}\,, \tag{1.281}$$

and prove it by induction, showing that this expression is consistent with the recursion relation.

Firstly, it is true for $n=1$, since

$$\begin{aligned} J^\mu(1^+) &= \frac{1}{\sqrt{2}} \frac{\langle q^- |\, \gamma_\mu \not{p}_1\, | q^+\rangle}{\langle q1\rangle\langle 1q\rangle} \\ &= \frac{1}{\sqrt{2}} \frac{\langle q^- |\, \gamma_\mu\, |1^-\rangle \cancel{\langle 1q\rangle}}{\langle q1\rangle \cancel{\langle 1q\rangle}} = \epsilon^\mu_+(p_1,q)\,. \end{aligned} \tag{1.282}$$

Let us suppose the formula is correct for $(n-1)$. Then we write the recursion relation, noting that $V_4^{\mu\nu\rho\sigma}$ and the $g^{\nu\rho}$ term in $V_3^{\mu\nu\rho}$ do not contribute, because they contract directly two currents, yielding terms like

$$\begin{aligned} &\left\langle q^- \middle| \gamma^\nu \not{P}_{1,i} \middle| q^+ \right\rangle \left\langle q^- \middle| \gamma_\nu \not{P}_{i+1,n} \middle| q^+ \right\rangle \\ &= -\left\langle q^- \middle| \not{P}_{1,i} \gamma^\nu \middle| q^+ \right\rangle \left\langle q^- \middle| \gamma_\nu \not{P}_{i+1,n} \middle| q^+ \right\rangle \\ &= -\lambda^a(q)(\overline{\sigma}_\mu)_{a\dot{a}}(\sigma^\nu)^{\dot{a}b}\lambda_b(q)\lambda^c(q)(\overline{\sigma}_\nu)_{c\dot{c}}(\sigma_\rho)^{\dot{c}d}\lambda_d(q) P^\mu_{1,i} P^\rho_{i+1,n} \\ &\text{Fierzing} \to 2\delta^{\dot{a}}_{\dot{c}}\delta^b_c \\ &= -2\lambda^a(q)(\overline{\sigma}_\mu)_{a\dot{a}}(\sigma_\rho)^{\dot{a}d}\lambda_d(q)\lambda^b(q)\lambda_b(q) P^\mu_{1,i} P^\rho_{i+1,n} \\ &= -2\left\langle q^- \middle| \not{P}_{1,i} \not{P}_{i+1,n} \middle| q^+ \right\rangle \cancelto{0}{\langle qq\rangle}. \end{aligned} \tag{1.283}$$

We can then write

$$
\begin{aligned}
& J^\mu(1^+,2^+,\ldots,n^+) \\
= \quad & -\frac{i}{P_{1,n}^2}\sum_{i=1}^{n-1}\frac{i}{\sqrt{2}}[2g^{\mu\rho}P_{i+1,n}^\nu - 2g^{\mu\nu}P_{1,i}^\rho] \\
& \cdot\frac{1}{\sqrt{2}}\frac{\langle q^-|\,\gamma_\nu \not{P}_{1,i}\,|q^+\rangle}{\langle q1\rangle\langle 12\rangle\cdots\langle (i-1)i\rangle\langle iq\rangle}\frac{1}{\sqrt{2}}\frac{\langle q^-|\,\gamma_\rho \not{P}_{i+1,n}\,|q^+\rangle}{\langle q(i+1)\rangle\cdots\langle (n-1)n\rangle\langle nq\rangle} \\
= \quad & \frac{1}{\sqrt{2}P_{1,n}^2}\frac{1}{\langle q1\rangle\langle 12\rangle\cdots\langle (n-1)n\rangle\langle nq\rangle}\sum_{i=1}^{n-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} \\
& \cdot\left(\left\langle q^-\right|\not{P}_{i+1,n}\not{P}_{1,i}\left|q^+\right\rangle\left\langle q^-\right|\gamma^\mu\not{P}_{i+1,n}\left|q^+\right\rangle - \left\langle q^-\right|\gamma^\mu\not{P}_{1,i}\left|q^+\right\rangle\left\langle q^-\right|\not{P}_{1,i}\not{P}_{i+1,n}\left|q^+\right\rangle\right) . \quad (1.284)
\end{aligned}
$$

Then we charge conjugate the current,

$$
\left\langle q^-\right|\not{P}_{1,i}\not{P}_{i+1,n}\left|q^+\right\rangle = -\left\langle q^-\right|\not{P}_{i+1,n}\not{P}_{1,i}\left|q^+\right\rangle\,, \tag{1.285}
$$

and add $\langle q^-|\,\not{P}_{i+1,n}\not{P}_{i+1,n}\,|q^+\rangle = P_{i+1,n}^2\,\langle q^-|\,|q^+\rangle = 0$. We can re-write the current as

$$
\begin{aligned}
& J^\mu(1^+,2^+,\ldots,n^+) \\
= \quad & \frac{1}{\sqrt{2}P_{1,n}^2}\frac{\langle q^-|\,\gamma^\mu\not{P}_{1,n}\,|q^+\rangle}{\langle q1\rangle\langle 12\rangle\cdots\langle (n-1)n\rangle\langle nq\rangle}\sum_{i=1}^{n-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle}\left\langle q^-\right|\not{P}_{i+1,n}\not{P}_{1,n}\left|q^+\right\rangle\,. \quad (1.286)
\end{aligned}
$$

Next, we use the identity (see app. H.22)

$$
\sum_{i=1}^{n-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle}\left\langle q^-\right|\not{P}_{i+1,n} = \frac{\langle 1^-|\,\not{P}_{1,n}}{\langle 1q\rangle}\,, \tag{1.287}
$$

based on the eikonal identity (see app. H.21)

$$
\sum_{i=j}^{k-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} = \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle}\,, \tag{1.288}
$$

and we get

$$
\begin{aligned}
J^\mu(1^+,2^+,\ldots,n^+) \quad &= \frac{1}{\sqrt{2}P_{1,n}^2}\frac{\langle q^-|\,\gamma^\mu\not{P}_{1,n}\,|q^+\rangle}{\langle q1\rangle\langle 12\rangle\cdots\langle (n-1)n\rangle\langle nq\rangle}\frac{\langle 1^-|\,\not{P}_{1,n}\not{P}_{1,n}\,|q^+\rangle}{\langle 1q\rangle} \\
&= \frac{\langle q^-|\,\gamma^\mu\not{P}_{1,n}\,|q^+\rangle}{\sqrt{2}\langle q1\rangle\langle 12\rangle\cdots\langle (n-1)n\rangle\langle nq\rangle}\,. \qquad (1.289)
\end{aligned}
$$

The amplitudes $iA_{n+1}^{tree}(1^+,2^+,\ldots,(n+1)^\pm)$ are obtained by multiplying by $iP_{1,n}^2$, by contracting with $\epsilon_\pm^\mu(P_{n+1})$ and taking the limit $P_{1,n}^2 = p_{n+1}^2 \to 0$. Since there is no $P_{1,n}^2$ pole in the current, we conclude that

$$
A_{n+1}^{tree}(1^+,2^+,\ldots,(n+1)^\pm) = 0\,, \tag{1.290}
$$

which is the third time that we prove it.

Also by induction, one can obtain (see app. H.23)

$$J^\mu(1^-, 2^+, \dots, n^+) = \frac{1}{\sqrt{2}} \frac{\langle 1^- | \gamma^\mu \not{P}_{2,n} | 1^+ \rangle}{\langle 12 \rangle \cdots \langle n1 \rangle} \sum_{k=3}^{n} \frac{\langle 1^- | \not{p}_k \not{P}_{1,k} | 1^+ \rangle}{P^2_{1,k-1} P^2_{1,k}} , \tag{1.291}$$

with polarisations $\epsilon^\mu_-(p_1, p_2)$ and $\epsilon^\mu_+(p_i, p_1)$, with $i = 2, \dots, n$.

In app. H.20, we show that $J^\mu(1^-, 2^+) = 0$, and

$$J^\mu(1^-, 2^+, 3^+) = \frac{1}{\sqrt{2}} \frac{\langle 1^- | \gamma^\mu \not{P}_{2,3} | 1^+ \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \frac{\langle 1^- | \not{p}_3 \not{P}_{1,3} | 1^+ \rangle}{P^2_{1,2} P^2_{1,3}} , \tag{1.292}$$

which is the first non-trivial case of this current, and agrees with the ansatz.

Then one assumes the ansatz to be correct for $J^\mu(1^-, 2^+, \dots, (n-1)^+)$, with $n \geq 4$. We sketch how the proof by induction unfolds, which is given app. H.23. One writes the recursion relation, noting as before that $V_4^{\mu\nu\rho\sigma}$ and the $g^{\nu\rho}$ term in $V_3^{\mu\nu\rho}$ do not contribute, because they contract directly two currents, yielding terms like $\langle 1^- | \gamma^\nu | 2^- \rangle \langle 1^- | \gamma_\nu \gamma_a | 1^+ \rangle$ and $\langle 1^- | \gamma^\nu \gamma_a | 1^+ \rangle \langle 1^- | \gamma_\nu \gamma_\beta | 1^+ \rangle$, which can all be Fierzed away.

The recursion relation is reduced to

$$\begin{aligned} J^\mu(1^-, 2^+, \dots, n^+) \;=\; & -\frac{i}{P^2_{1,n}} \Big[V_3^{\mu\nu\rho}(p_1, P_{2,n}) J_\nu(1^-) J_\rho(2^+, \dots, n^+) \\ & + \sum_{i=3}^{n-1} V_3^{\mu\nu\rho}(P_{1,i}, P_{i+1,n}) J_\nu(1^-, \dots, i^+) J_\rho((i+1)^+, \dots, n^+) \Big] . \end{aligned} \tag{1.293}$$

with $V_3^{\mu\nu\rho}(P, Q) = \frac{i}{\sqrt{2}} [2 g^{\mu\rho} Q^\nu - 2 g^{\mu\nu} P^\rho]$.

From $J^\mu(1^-, 2^+, \dots, n^+)$, the amplitude $i A^{tree}_{n+1}(1^-, 2^+, \dots, (n+1)^-)$ is obtained by multiplying by $i P^2_{1,n}$, by contracting with $\epsilon^\mu_-(p_{n+1})$, and taking the limit $P^2_{1,n} = p^2_{n+1} \to 0$. The only $P^2_{1,n}$ pole in eq. (1.292) is for $k = n$. For the polarisation of gluon $(n+1)$, we use $\epsilon^\mu_-(p_{n+1}, p_n)$. Then the amplitude is

$$\begin{aligned} & i A^{tree}_{n+1}(1^-, 2^+, \dots, n^+, (n+1)^-) \\ & = \; -i \frac{\langle n^+ | \gamma^\mu | (n+1)^+ \rangle}{\sqrt{2} [n(n+1)]} \frac{1}{\sqrt{2}} \frac{\langle 1^- | \gamma_\mu \not{P}_{1,n} | 1^+ \rangle}{\langle 12 \rangle \cdots \langle n1 \rangle} \frac{\langle 1^- | \not{p}_n \not{P}_{1,n} | 1^+ \rangle}{P^2_{1,n-1}} . \end{aligned} \tag{1.294}$$

We need to perform the contraction,

$$\begin{aligned} & \left\langle n^+ \middle| \gamma^\mu \middle| (n+1)^+ \right\rangle \left\langle 1^- \middle| \gamma_\mu \gamma_\nu \middle| 1^+ \right\rangle \\ & \quad = \xi^\dagger_-(1) \overline{\sigma}_\mu \sigma_\nu \xi_+(1) \xi^+_+(n) \sigma^\mu \xi_+(n+1) \\ & \quad = \lambda^b(1) (\overline{\sigma}_\mu)_{b\dot{b}} (\sigma_\nu)^{\dot{b}c} \lambda_c(1) \tilde{\lambda}_{\dot{a}}(n) (\sigma^\mu)^{\dot{a}a} \lambda_a(n+1) , \end{aligned} \tag{1.295}$$

then we use the Fierz identity

$$
\begin{aligned}
&= 2\lambda^a(1)\lambda_a(n+1)\tilde{\lambda}_{\dot{a}}(n)(\sigma_\nu)^{\dot{a}c}\lambda_c(1) \\
&= 2\,\langle 1(n+1)\rangle \left\langle n^+\right|\gamma_\nu\left|1^+\right\rangle\,. \qquad (1.296)
\end{aligned}
$$

Further, through momentum conservation, we replace $\not{P}_{1,n} = -\not{p}_{n+1}$ and $P^2_{1,n-1} = s_{n,n+1}$, so

$$
\begin{aligned}
&A^{tree}_{n+1}(1^-, 2^+, \ldots, n^+, (n+1)^-) \\
&= -\frac{\langle 1(n+1)\rangle}{\langle 12\rangle\cdots\langle n1\rangle}\frac{\langle n^+|\not{p}_{n+1}|1^+\rangle\langle 1^-|\not{p}_n\not{p}_{n+1}|1^+\rangle}{[n(n+1)]s_{n,n+1}} \\
&= -\frac{\langle 1(n+1)\rangle}{\langle 12\rangle\cdots\langle n1\rangle}\frac{[n(n+1)]\langle(n+1)1\rangle\langle 1n\rangle[n(n+1)]\langle(n+1)1\rangle}{[n(n+1)]^2\langle(n+1)n\rangle} \\
&= -\frac{\langle 1(n+1)\rangle^3}{\langle 12\rangle\cdots\langle n(n+1)\rangle} \\
&= \frac{\langle 1(n+1)\rangle^4}{\langle 12\rangle\cdots\langle n(n+1)\rangle\langle(n+1)1\rangle}\,, \qquad (1.297)
\end{aligned}
$$

which proves Parke-Taylor formula.

In the literature, one can find also in closed form the current $J^\mu(1^-, 2^-, 3^+, \ldots, n^+)$ from which the NMHV amplitudes $A(--+\cdots+-)$ with three adjacent negative-helicity gluons can be derived [24], however the greatest value of the Berends-Giele recursion relations is to provide an efficient (in fact, still the fastest) method to generate numerically the helicity amplitudes.

1.10 Amplitudes with Photons

Amplitudes with photons can be obtained from the amplitude of $q\bar{q} \to (n-2)$ gluons (1.173) by replacing the $SU(N_c)$ generator of a gluon, say gluon n, with the $U(1)$ generator $(T^{a_{U(1)}})_i^{\,j} = \frac{1}{\sqrt{N_c}}\delta_i^j$. Since the $U(1)$ generator commutes with $SU(N_c)$, we obtain the amplitude,

$$
\begin{aligned}
&M^{tree}_n(1^{\lambda_1}_{\bar{q}}, 2^{-\lambda_1}_q, 3_g, \ldots, (n-1)_g, n_\gamma) \\
&\quad = \sqrt{2}Q_q e g^{n-3} \sum_{\sigma\in S_{n-3}} (T^{a_{\sigma_3}}\cdots T^{a_{\sigma_{n-1}}})^{i_2}_{i_1} A^{tree}_n(1^{\lambda_1}_{\bar{q}}, 2^{-\lambda_1}_q, \sigma_3, \ldots, \sigma_{n-1}, n_\gamma)\,. \qquad (1.298)
\end{aligned}
$$

where a factor of the strong coupling g has been replaced by the QED coupling $\sqrt{2}Q_q e$, and the sub-amplitude is obtained from $A^{tree}_n(1_{\bar{q}}, 2_q, 3, \ldots, n)$ by summing over the positions of gluon n,

$$
\begin{aligned}
&A^{tree}_n(1_{\bar{q}}, 2_q, 3, \ldots, n-1, n_\gamma) = A^{tree}_n(1_{\bar{q}}, 2_q; 3, \ldots, n) \\
&\quad + \quad A^{tree}_n(1_{\bar{q}}, 2_q; 3, \ldots, n, n-1) + \ldots + A^{tree}_n(1_{\bar{q}}, 2_q; n, 3, \ldots, n-1)\,. \qquad (1.299)
\end{aligned}
$$

In particular, for the MHV configuration (1.184), where i is the negative-helicity gluon or photon, we write

$$\begin{aligned}
&A_n^{tree}(1_{\bar{q}}^+, 2_q^-, 3^+, \ldots, i^-, \ldots, n_\gamma^+) = \frac{\langle 2i\rangle^3\langle 1i\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle} \\
&\quad + \frac{\langle 2i\rangle^3\langle 1i\rangle}{\langle 12\rangle\cdots\langle n(n-1)\rangle\langle (n-1)1\rangle} + \ldots + \frac{\langle 2i\rangle^3\langle 1i\rangle}{\langle 12\rangle\langle 2n\rangle\langle n3\rangle\cdots\langle (n-1)1\rangle} \\
&= \frac{\langle 2i\rangle^3\langle 1i\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)1\rangle}\Big[\frac{\langle (n-1)1\rangle}{\langle (n-1)n\rangle\langle n1\rangle} + \frac{\langle (n-2)(n-1)\rangle}{\langle (n-2)n\rangle\langle n(n-1)\rangle} + \cdots + \frac{\langle 23\rangle}{\langle 2n\rangle\langle n3\rangle}\Big] \\
&= \frac{\langle 2i\rangle^3\langle 1i\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)1\rangle}\sum_{i=2}^{n-1}\frac{\langle i(i+1)\rangle}{\langle in\rangle\langle n(i+1)\rangle}\,. \qquad (1.300)
\end{aligned}$$

then we use the eikonal identity (1.288) and we obtain

$$A_n^{tree}(1_{\bar{q}}^+, 2_q^-, 3^+, \ldots, i^-, \ldots, n_\gamma^+) = \frac{\langle 21\rangle}{\langle 2n\rangle\langle n1\rangle}\frac{\langle 2i\rangle^3\langle 1i\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)1\rangle}\,. \qquad (1.301)$$

The procedure can be iterated: the amplitude with r gluons and m photons, with $m+r=n-2$, can be written as

$$\begin{aligned}
&M_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, 3_g, \ldots, (r+2)_g, (r+3)_\gamma, \ldots, n_\gamma) \\
&= (\sqrt{2}Q_q e)^m g^r \sum_{\sigma\in S_r}(T^{a_{\sigma_3}}\cdots T^{a_{\sigma_r}})_{i_1}^{i_2} A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, \sigma_3, \ldots, \sigma_{r+2}, (r+3)_\gamma, \ldots, n_\gamma)\,, \qquad (1.302)
\end{aligned}$$

where

$$A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, 3, \ldots, r+2, (r+3)_\gamma, \ldots, n_\gamma) = \sum_{\sigma\in S_{n-2}/S_r} A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, \sigma(3), \ldots, \sigma(n))\,. \qquad (1.303)$$

In particular, for the MHV configuration $(--+\cdots+)$, iterating the eikonal identity (1.288), we obtain

$$A_n^{tree}(1_{\bar{q}}^+, 2_q^-, 3, \ldots, r+2, (r+3)_\gamma, \ldots, n_\gamma) = \frac{\langle 2i\rangle^3\langle 1i\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle (r+2)1\rangle}\prod_{j=r+3}^{n}\frac{\langle 21\rangle}{\langle 2j\rangle\langle j1\rangle}\,, \qquad (1.304)$$

with i the negative-helicity gluon or photon.

Finally, the amplitude with $q\bar{q}\to(n-2)$ photons is

$$M_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, 3_\gamma, \ldots, n_\gamma) = (\sqrt{2}Q_q e)^{n-2} A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, 3_\gamma, \ldots, n_\gamma)\,, \qquad (1.305)$$

where

$$A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, 3_\gamma, \ldots, n_\gamma) = \sum_{S_{n-2}} A_n^{tree}(1_{\bar{q}}^{\lambda_1}, 2_q^{-\lambda_1}, \sigma_3, \ldots, \sigma_n)\,. \qquad (1.306)$$

In particular, for the MHV configuration, we obtain

$$A_n^{tree}(1_{\bar{q}}^+, 2_q^-, 3_\gamma^+, \ldots, i_\gamma^-, \ldots, n_\gamma^+) = \frac{\langle 2i \rangle^3 \langle 1i \rangle}{\langle 12 \rangle \langle 21 \rangle} \prod_{j=3}^{n} \frac{\langle 21 \rangle}{\langle 2j \rangle \langle j1 \rangle} \,. \tag{1.307}$$

1.11 A collider physics summary

Let us take stock of what we have done so far. Using the spinor helicity formalism, we have computed the (squared) amplitudes for:

- $$e^+e^- \to \mu^+\mu^-.$$
- $$e^+e^- \to q\bar{q} \xrightarrow{\text{crossing}} \begin{cases} qe \to qe : \text{DIS} \\ q\bar{q} \to \ell^+\ell^- : \text{Drell} - \text{Yan} \end{cases} \tag{1.308}$$
- $$e^+e^- \to \gamma\gamma.$$
- $$e^+e^- \to q\bar{q}g : \begin{cases} 3 - \text{jet production in } e^+e^- \text{ (relevant for } \mathrm{a_s}) \\ \text{soft and collinear limits} \end{cases} \tag{1.309}$$
- $$q\bar{q} \to (n-2) \text{ gluons} : \begin{cases} \text{colour decomposition} \\ \text{MHV amplitudes} \end{cases} \tag{1.310}$$
- $$q\bar{q} \to gg$$
- $$gg \to (n-2) \text{ gluons} : \begin{cases} \text{colour decompositions} \\ \text{MHV amplitudes} \\ \text{multi} - \text{Regge kinematics} \\ \text{BCJ relations} \end{cases} \tag{1.311}$$
- $gg \to gg$
- $q\bar{q} \to r$ gluons $+\ m$ photons, with $m + r = n - 2$.
- $q\bar{q} \to q'\bar{q}'$
- $q\bar{q} \to q\bar{q}$
- $gg \to H$ (HEFT)
- $gg \to Hg$ (HEFT): soft and collinear limits
- supersymmetric Ward identities

- Berends - Giele recursion relations

This sums up the knowledge of tree-level helicity amplitudes at the turn of the new millennium (except that the BCJ relations were found in 2008, and that we have not covered $gg \to gggg$ and NMHV amplitudes. We will do it with BCFW) and provides all the basic processes of e^+e^- colliders and hadron colliders.

At the end of 2003, Witten jump-starts a new thread of studies of amplitudes, which is still ongoing. In a work on twistor string theory [26], Witten studies amplitudes in a $(++--)$ metric, i.e a metric with two time components. Real momenta in a $(++--)$ are equivalent to complex momenta in the usual $(+---)$ metric. As we will see, that has far-reaching implications.

1.12 Complex momenta

1.12.1 Three-particle kinematics

In sec. 1.5.3, we discussed the collinear limit in $e^+e^- \to q\bar{q}g$ scattering, fig 1.8. We considered the limit as gluon 4 becomes collinear to quark 3, $p_3 || p_4$. We said that the momentum $P = p_3 + p_4$ of the quark parent goes on-shell as $P^2 = 2p_3 \cdot p_4 \to 0$. Let us investigate the kinematics a bit more precisely. Let us suppose that $P^\mu = (P^0, 0, 0, P^2)$ is aligned with the beam direction $p^\mu = (p/2, 0, 0, p/2)$. In light-cone coordinates, they are

$$p^\mu = (p, 0; 0_\perp), \qquad P^\mu = (P^+, P^-; 0_\perp) \,. \tag{1.312}$$

Fixing the minus light-cone direction as $\eta^\mu = (0, \eta^-; 0_\perp)$ and purely transverse vectors as $k_\perp^\mu = (0, 0; k_\perp)$, with $k_\perp^2 = -\left|\vec{k}_\perp\right|^2 = -k_\perp k_\perp^*$, for $k_\perp = k_x + ik_y$, we may use the Sudakov (or light-cone) parametrisation and write p_3 and p_4 as

$$\begin{aligned} p_3^\mu &= zp^\mu + k_{3\perp}^\mu - \frac{k_{3\perp}^2}{z} \frac{\eta^\mu}{2p\cdot\eta} \,, \\ p_4^\mu &= (1-z)p^\mu + k_{4\perp}^\mu - \frac{k_{4\perp}^2}{1-z} \frac{\eta^\mu}{2p\cdot\eta} \,. \end{aligned} \tag{1.313}$$

Using the fact that $p^2 = \eta^2 = 0$ and $p \cdot k_{i\perp} = \eta \cdot k_{i\perp} = 0$, we can easily check that $p_3^2 = k_{3\perp}^2 - k_{3\perp}^2 \frac{2zp\cdot\eta}{z2p\cdot\eta} = 0$ and likewise for p_4^2, i.e. p_3 and p_4 are on the mass shell. Further, momentum conservation $P = p_3 + p_4$ implies that $k_{3\perp} = -k_{4\perp} \equiv k_\perp$. Then

$$P^\mu = p^\mu - \Big(\frac{k_\perp^2}{z} + \frac{k_\perp^2}{1-z}\Big) \frac{\eta^\mu}{2p\cdot\eta} \,. \tag{1.314}$$

so

$$P^2 = -\Big(\frac{k_\perp^2}{z} + \frac{k_\perp^2}{1-z}\Big) = -\frac{k_\perp^2}{z(1-z)} \,, \tag{1.315}$$

i.e. P is time-like ($P^2 > 0$ since $k_\perp^2 < 0$) and goes on-shell as $k_\perp \to 0$ (note that if one of the emitted particles is soft, $z \to 0$ or $(1-z) \to 0$, then $k_\perp$ and z or $(1-z)$ go to zero at the same rate, such that P^2 still vanishes). This is not unexpected: $P^2 > 0$ implies that the parent particle is either off-shell or massive. A massless parent particle can be on-shell only in in the strict collinear limit, $k_\perp = 0$, where all three particles are collinear. Thus, a massless 3-particle scattering, with $p_1^\mu + p_2^\mu + p_3^\mu = 0$ and $p_1^2 = p_2^2 = p_3^2 = 0$ is impossible, because it implies that $s_{12}^2 = (p_1 + p_2)^2 = p_3^2 = 0$, and likewise $s_{13} = s_{23} = 0$. Accordingly, $\langle ij\rangle = [ij] = 0$, with $i, j = 1, 2, 3$. The only solution, as we saw, is that the three particles are collinear.

The obvious assumption above is that momenta are real. However, if momenta are complex, there is a way out: for complex momenta, $\tilde{\lambda}_{\dot{a}}$ is not the hermitian conjugate of λ_a. Accordingly, $[pk]$ is not the complex conjugate of $\langle kp\rangle$, although $s_{pk} = 2p \cdot k = \langle pk\rangle[kp]$ still holds (we introduced the notion of squaring through projection operators (1.36) or Fierz and Gordon identities (1.81), without using complex conjugation).

1.12.2 Constraining three-particle amplitudes

For a massless three-particle scattering, momentum conservation is $p_1^\mu + p_2^\mu + p_3^\mu = 0$, or using eq. (1.55),

$$|1^\pm\rangle\langle 1^\pm| + |2^\pm\rangle\langle 2^\pm| + |3^\pm\rangle\langle 3^\pm| = 0\,. \tag{1.316}$$

With complex momenta, two chirally conjugate solutions exist: multiplying eq. (1.316) by $\langle 1^-|$ or $\langle 2^-|$, we get

$$\langle 12\rangle\langle 2^+| + \langle 13\rangle\langle 3^+| \;=\; 0\,, \tag{1.317}$$

$$\langle 21\rangle\langle 1^+| + \langle 23\rangle\langle 3^+| \;=\; 0\,, \tag{1.318}$$

then either $\langle 12\rangle = 0$, which implies that $\langle 13\rangle = \langle 23\rangle = 0$, or $\langle 1^+| \propto \langle 2^+| \propto \langle 3^+|$, i.e. $\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$, which implies that $[12] = [13] = [23] = 0$, i.e.

$$i)\quad \tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3 \quad\Rightarrow\quad [ij] = 0 \quad \text{and} \quad s_{ij} = 0\,, \quad \text{but} \quad \langle ij\rangle \neq 0\,, \tag{1.319}$$

for $i, j = 1, 2, 3$. Likewise, one shows that

$$ii)\quad \lambda_1 \propto \lambda_2 \propto \lambda_3 \quad\Rightarrow\quad \langle ij\rangle = 0 \quad \text{and} \quad s_{ij} = 0\,, \quad \text{but} \quad [ij] \neq 0\,. \tag{1.320}$$

Cases i) and ii) describe just two points in phase space, related to each other by parity. So with complex momenta a solution exists, but it must consists of functions of either right-handed spinors $\langle ij\rangle$ or left-handed spinors $[ij]$, but not functions of both.

1.12.3 Little group scaling

In the Tutorials (app. H.26), we compute:
i) the MHV three-point amplitude,

$$iA_3^{tree}(1^-, 2^-, 3^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,. \tag{1.321}$$

ii) the $\overline{\text{MHV}}$ three-point amplitude,

$$iA_3^{tree}(1^+, 2^+, 3^-) = -i\frac{[12]^4}{[12][23][31]}\,. \tag{1.322}$$

through the three-gluon vertex, but there is a deeper way to compute the dependence of three-point amplitudes on the spinor products. We know that the momentum $p_{a\dot{a}} = \lambda_a(p)\tilde{\lambda}_{\dot{a}}(p)$ or $p^{\dot{a}a} = \tilde{\lambda}^{\dot{a}}(p)\lambda^a(p)$, or otherwise $\not{p} = |p^-\rangle\langle p^-| + |p^+\rangle\langle p^+|$, is invariant under little group scaling (1.66), so that the spinors scale as in eq. (1.72),

$$\begin{cases} \lambda_a(p) \sim \xi_+(p) \sim \left|p^+\right\rangle \sim |\;\rangle \to z\lambda_a(p)\,, \\ \lambda^a(p) \sim \xi_-^\dagger(p) \sim \left\langle p^-\right| \sim \langle\;| \to z\lambda^a(p)\,, \end{cases} \quad \text{in} \quad (0, 1/2) \tag{1.323}$$

$$\begin{cases} \tilde{\lambda}^{\dot{a}}(p) \sim \xi_-(p) \sim \left|p^-\right\rangle \sim |\;] \to z^{-1}\tilde{\lambda}^{\dot{a}}(p)\,, \\ \tilde{\lambda}_{\dot{a}}(p) \sim \xi_+^\dagger(p) \sim \left\langle p^+\right| \sim [\;| \to z^{-1}\tilde{\lambda}_{\dot{a}}(p)\,, \end{cases} \quad \text{in} \quad (1/2, 0)\,, \tag{1.324}$$

with $z \in \mathbb{C}$. Note that λ_a and λ^a, i.e. the right-handed spinors, scale in the same way since they are related by charge conjugation, $\lambda_a = \epsilon_{ab}\lambda^b$. Likewise for the left-handed spinors, $\tilde{\lambda}^{\dot{a}}$ and $\tilde{\lambda}_{\dot{a}}$.

Under little group scaling of the spinors associated to the gluon momentum, the polarisation of a positive-helicity gluon,

$$\epsilon_\mu^{+*}(p, k) = \frac{\langle k^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\langle k^-|p^+\rangle} = \frac{\lambda^a(k)(\bar{\sigma}_\mu)_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(p)}{\sqrt{2}\lambda^b(k)\lambda_b(p)}\,,$$

scales like

$$\epsilon_\mu^{+*}(p, k) \to z^{-2}\,\epsilon_\mu^{+*}(p, k)\,. \tag{1.325}$$

Likewise,

$$\epsilon_\mu^{-*}(p, k) = -\frac{\langle k^+|\,\gamma_\mu\,|p^+\rangle}{\sqrt{2}\langle k^+|p^-\rangle}\,,$$

scales like

$$\epsilon_\mu^{-*}(p, k) \to z^2\,\epsilon_\mu^{-*}(p, k)\,. \tag{1.326}$$

Note that the polarisation is invariant under scaling of the spinors associated to the reference vector k. The scaling above is consistent with the scaling $\epsilon_\mu^+(p, k) \to e^{i\phi}\epsilon_\mu^+(p, k)$ of a right-handed spin-1 particle, we saw in sec. 1.4.

Every positive (negative)-helicity gluon brings one more factor z^{-2} (z^2) to the scaling of the amplitude. Gluon amplitudes are written in terms of the spinor products associated to the momenta of gluons and reference vectors, which though are little group invariants. So the little group scaling of an amplitude is determined entirely by the gluon polarisations,

$$A(1,\ldots,n) \to \prod_{i=1}^{n} z^{-2h_i} A(1,\ldots,n)\,, \tag{1.327}$$

where h_i is the helicity of the i^{th} particle.

Suppose a three-point amplitude is made of right-handed spinor products $\langle ij\rangle$,

$$A_3^{tree}(1^{h_1},2^{h_2},3^{h_3}) \propto \langle 12\rangle^{x_{12}}\langle 23\rangle^{x_{23}}\langle 13\rangle^{x_{13}}\,, \tag{1.328}$$

with $\{h_1,h_2,h_3\} = \pm 1$. Then under little group scaling,

$$x_{12}+x_{13} = -2h_1\,, \qquad x_{12}+x_{23} = -2h_2\,, \qquad x_{13}+x_{23} = -2h_3\,, \tag{1.329}$$

which have solutions,

$$x_{12} = -h_1-h_2+h_3\,, \qquad x_{13} = -h_1-h_3+h_2\,, \qquad x_{23} = -h_2-h_3+h_1\,, \tag{1.330}$$

such that

$$A_3^{tree}(1^-,2^-,3^+) \propto \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}\,, \tag{1.331}$$

so little group scaling fixes uniquely the dependence of $A_3^{tree}(1^-,2^-,3^+)$ on the right-handed spinor products.

Now, if we suppose that the three-point amplitude is made of left-handed spinor products,

$$A_3^{tree}(1^{h_1},2^{h_2},3^{h_3}) \propto [12]^{y_{12}}[23]^{y_{23}}[13]^{y_{13}}\,, \tag{1.332}$$

then under little group scaling,

$$y_{12}+y_{13} = 2h_1\,, \qquad y_{12}+y_{23} = 2h_2\,, \qquad y_{13}+y_{23} = 2h_3\,, \tag{1.333}$$

which have solutions,

$$y_{12} = h_1+h_2-h_3\,, \qquad y_{13} = h_1+h_3-h_2\,, \qquad y_{23} = h_2+h_3-h_1\,, \tag{1.334}$$

such that

$$A_3^{tree}(1^+,2^+,3^-) \propto \frac{[12]^3}{[13][23]}\,. \tag{1.335}$$

Note that if we tried to fit $A_3^{tree}(1^-,2^-,3^+)$ with left-handed spinor products, eq. (1.334) would imply that

$$A_3^{tree}(1^-,2^-,3^+) \propto \frac{[13][23]}{[12]^3}\,, \tag{1.336}$$

but this would have the wrong mass dimension. In fact, $A_3^{tree}(1^{h_1},2^{h_2},3^{h_3})$ must have mass dimension $=1$, since it comes from the three-gluon vertex which has a momentum in the numerator (this can also be seen from an n-boson function, whose mass dimension is $4-nB$, where B is the boson field), but $\langle,\rangle$ and $[,]$ have mass dimension $=1$, so $A_3^{tree}(1^{h_1},2^{h_2},3^{h_3})$ must have one more spinor product in the numerator.

We conclude that by little group scaling and dimensional analysis, the three-point MHV amplitudes are constrained to the form given in eqs. (1.331) and (1.335).

By Lorentz invariance, the three-particle amplitude is only restricted to be a function of $\langle ij\rangle$ and $[ij]$. In order not to vanish in either point i) or ii) it cannot be a function simultaneously of $\langle ij\rangle$ and $[ij]$. So we can write that in general,

$$M_3 = M_3^H\big(\langle 12\rangle,\langle 23\rangle,\langle 13\rangle\big) + M_3^A\big([12],[23],[13]\big)\,, \tag{1.337}$$

where M_3^H and M_3^A are generic functions of the spinor products (H and A refer to"holomorphic" and "antiholomorphic"). Then we argued that if the holomorphic part is made of right-handed spinor products (1.328), and the antiholomorphic part is made of left-handed spinor products (1.332),

$$\begin{aligned} M_3^H\big(\langle 12\rangle,\langle 23\rangle,\langle 13\rangle\big) &\propto \langle 12\rangle^{x_{12}}\langle 23\rangle^{x_{23}}\langle 13\rangle^{x_{13}}\,,\\ M_3^A\big([12],[23],[13]\big) &\propto [12]^{y_{12}}[13]^{y_{13}}[23]^{y_{23}}\,, \end{aligned}$$

little group scaling implies eqs. (1.330) and (1.334), i.e. that

$$\begin{cases} x_{12} = -y_{12} = -h_1 - h_2 + h_3\,,\\ x_{13} = -y_{13} = -h_1 - h_3 + h_2\,,\\ x_{23} = -y_{23} = -h_2 - h_3 + h_1\,, \end{cases}$$

and that in Yang-Mills theory, for the helicity configuration $(1^-,2^-,3^+)$, M_3^A would not contribute to $A_3(1^-,2^-,3^+)$ because it would have the wrong mass dimension. Of course, the assumption is that in Yang-Mills theory, the coupling is dimensionless. Let us relax this assumption. In a generic theory,

$$M_3^H\big(\langle 12\rangle,\langle 23\rangle,\langle 13\rangle\big) = \kappa_{abc}^H\langle 12\rangle^{x_{12}}\langle 23\rangle^{x_{23}}\langle 13\rangle^{x_{13}}\,, \tag{1.338}$$

$$M_3^A\big([12],[23],[13]\big) = \kappa_{abc}^A[12]^{y_{12}}[13]^{y_{13}}[23]^{y_{23}}\,. \tag{1.339}$$

where k_{abc}^H and k_{abc}^A are couplings, eventually dimensionful, in which we include internal degrees of freedom, like colour.

Now, the three-particle amplitude M_3 must have the correct physical behaviour for real momenta, i.e. it must vanish when both $\langle ij\rangle$ and $[ij]$ go to zero. Since from eqs. (1.330) and (1.334),

$$x_{12} + x_{13} + x_{23} = -(y_{12} + y_{13} + y_{23}) = -(h_1 + h_2 + h_3)\,, \tag{1.340}$$

for $h_1 + h_2 + h_3 < 0$ we must set $k^A_{abc} = 0$ in order for M^A_3 not to blow up. Conversely, for $h_1 + h_2 + h_3 > 0$, we must set $k^H_{abc} = 0$. So we can write that in general,

$$M^H_3 = \kappa^H_{abc}\,\langle 12\rangle^{-h_1-h_2+h_3}\langle 13\rangle^{-h_1-h_3+h_2}\langle 23\rangle^{-h_2-h_3+h_1}\Theta(-h_1 - h_2 - h_3)\,, \tag{1.341}$$

$$M^A_3 = \kappa^A_{abc}\,[12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[13]^{h_1+h_3-h_2}\Theta(h_1 + h_2 + h_3)\,. \tag{1.342}$$

The only case we are excluding in this treatment is $h_1 + h_2 + h_3 = 0$.

Then, for a theory of several massless particles of a given integer spin s, we can replace $h = \pm s$, and eqs. (1.341) and (1.342) have each two solutions,

$$M^H_3(1^-_a, 2^-_b, 3^+_c) = \kappa^H_{abc}\,\Big(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\Big)^s\,, \tag{1.343}$$

$$M^H_3(1^-_a, 2^-_b, 3^-_c) = \kappa^H_{abc}\,\big(\langle 12\rangle\langle 23\rangle\langle 31\rangle\big)^s\,, \tag{1.344}$$

$$M^A_3(1^+_a, 2^+_b, 3^-_c) = \kappa^A_{abc}\,\Big(\frac{[12]^3}{[23][31]}\Big)^s\,, \tag{1.345}$$

$$M^A_3(1^+_a, 2^+_b, 3^+_c) = \kappa^A_{abc}\,\big([12][23][31]\big)^s\,, \tag{1.346}$$

where we have labelled particles also with their internal degrees of freedom, e.g. colour.

Since s is integer, the solutions above must be Bose symmetric. Since the spinor products are antisymmetric, this implies that for odd s, $\kappa^{H,A}_{abc}$ must be totally antisymmetric under the exchange of any two indices. Therefore, a theory of less than three massless particles of odd spin must have a trivial three-particle amplitude and, assuming that any amplitude can be constructed out of three-particle amplitudes, a trivial S-matrix.

1.12.4 Uniqueness of Yang-Mills

In this section, we follow sec 27.5 of ref. [5]. Firstly, we recall a general non-perturbative result, which is a consequence of unitarity (see sec. 10.2 of ref. [16] or sec 24.3 of ref. [5]): in a unitary theory, poles of Green's functions, and so of amplitudes, correspond to the exchange of on-shell intermediate states,

$$\lim_{P^2_{1,k}\to M^2} G_n(p_1, \dots, p_n) = (2\pi)^4\delta^4(\textstyle\sum p)\dfrac{i}{P^2_{1,k} - M^2 + i\epsilon}M^{1,k}(M^{k+1,n})^\dagger + \dots\,, \tag{1.347}$$

with $P_{1,k} = p_1 + \ldots + p_k$. All that is needed to prove it is that a one-particle state $|\psi\rangle$ transforms according to an irreducible representation (irrep) of the Poincaré group, so its momentum p^μ may go on-shell, with $p^2 = m^2$. We have been using this result, in the form of multi-particle factorisation, since sec. 1.5.3.

In sec. 1.12.3, we have concluded that self-interacting massless particles of odd spin s are only allowed if there are at least three particles with a fully antisymmetric coupling. Let us examine what additional constraints the pole structure of four-point tree amplitudes entails, by considering the four-gluon amplitude in Yang-Mills theory. The only thing that we will assume are multi-particle factorisation and complex momenta, so we can build four-point amplitudes out of three-point amplitudes. We consider the amplitude $M(1^-2^-3^+4^+)$. Little group scaling implies that

$$M(1234) \to t^{-2(h_1+h_2+h_3+h_4)} M(1234)\,, \tag{1.348}$$

and thus

$$M(1_a^- 2_b^- 3_c^+ 4_d^+) \to \langle 12\rangle^2 [34]^2 F^{abcd}(s,t,u)\,, \tag{1.349}$$

where a, b, c, d are the colour indices. Since an amplitude has mass dimensions, $4 - \frac{3}{2}\psi - G$, where ψ is the number of spin-1/2 fields and G is the number of spin-1 fields, $M(1234)$ is dimensionless, while the mass dimension of F is $[F] = [M]^{-4}$.

On the pole in the s-channel, the amplitude factorises as

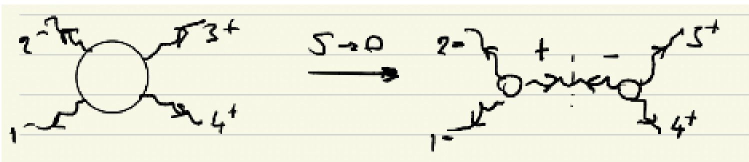


Figure 1.24: s-channel pole factorisation of four-gluon amplitude.

The intermediate state $P = -(p_1 + p_2) = p_3 + p_4$ goes on-shell with positive helicity in order for $M(1^-2^-P)$ not to vanish. Then

$$\lim_{s\to 0}\, iM(1^-2^-3^+4^+) = \frac{i\langle 12\rangle^3}{\langle 2P\rangle\langle P1\rangle}\,\frac{i}{s}\,\frac{-i[34]^3}{[3(-P)][(-P)4]}\; f^{abe}\, f^{ecd}\,. \tag{1.350}$$

Then we analytically continue [k(-P)] = i [kP] and use $\langle 2P\rangle[P4] = -\langle 21\rangle[14]$ and $[3P]\langle P4\rangle = [34]\langle 41\rangle$, so

$$\lim_{s\to 0}\, iM(1^-2^-3^+4^+) = \frac{-i}{s}\,\frac{\langle 12\rangle^{\cancel{3}\,2}[34]^{\cancel{3}\,2}}{\cancel{\langle 12\rangle}[14]\cancel{[34]}\langle 41\rangle}\; f^{abe}\, f^{ecd}\,. \tag{1.351}$$

Since $\langle 41\rangle[14] = s_{14} = s_{23} = t$, we have

$$\lim_{s\to 0}\, F^{abcd}(s,t,u) = -\frac{1}{st}\, f^{abe}\, f^{ecd}\,. \tag{1.352}$$

Note that $P^2 = 0 = \langle 12\rangle[21] = \langle 34\rangle[43]$ and consistently the three-particle amplitudes imply that $[12] = \langle 34\rangle = 0$.

On the pole in the t channel, the amplitude factorises as

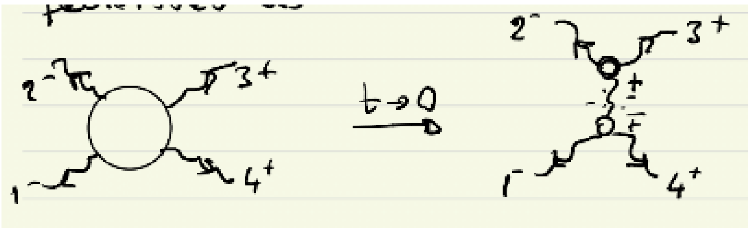


Figure 1.25: t-channel pole factorisation of four-gluon amplitude.

Both helicities contribute to the intermediate state P,

$$\begin{aligned}\lim_{t\to0}\, iM(1^-2^-3^+4^+) &= \left[\frac{i\langle P2\rangle^3}{\langle23\rangle\langle3P\rangle}\,\frac{i}{t}\,\frac{i[P4]^3}{[41][1P]} + \frac{i[3P]^3}{[23][P2]}\,\frac{i}{t}\,\frac{i\langle1P\rangle^3}{\langle41\rangle\langle P4\rangle}\right] f^{dae}\, f^{ebc} \\ &= \frac{i}{t}\left(\underbrace{\frac{\langle P2\rangle^3[P4]^3}{\langle23\rangle\langle3P\rangle[41][1P]}}_{1)} + \underbrace{\frac{[3P]^3\langle1P\rangle^3}{[23][P2]\langle41\rangle\langle P4\rangle}}_{2)}\right) f^{ade}\, f^{ebc}\,. \quad (1.353)\end{aligned}$$

Since,

$$P = p_1 + p_4 = -(p_2+p_3) \Rightarrow P^2 = 0 = \langle14\rangle[41] = \langle23\rangle[32]\,, \quad (1.354)$$

then

for	$[23] = \langle14\rangle = 0$,	only	1)	contributes,
for	$\langle23\rangle = [14] = 0$,	only	2)	contributes.

We use $\langle1P\rangle[P3] = \langle14\rangle[43]$ and $\langle2P\rangle[P4] = \langle21\rangle[14]$, and rewrite the two terms of eq. (1.353) as

$$1) \;=\; \frac{[14]^{\cancel{3}}\langle12\rangle^3}{\langle23\rangle\cancel{[41]^2}\langle43\rangle} = \langle12\rangle^2[34]^2\frac{\langle12\rangle[14]}{\langle23\rangle\langle43\rangle[34]^2} = \frac{\langle12\rangle^2[34]^2}{s}\frac{\cancel{(-)\langle21\rangle[14]}}{\cancel{\langle23\rangle[34]}}\,, \quad (1.355)$$

$$2) \;=\; \frac{\langle14\rangle^{\cancel{3}}[34]^3}{[23]\cancel{\langle41\rangle^2}[21]} = \langle12\rangle^2[34]^2\frac{\langle14\rangle[34]}{[23][21]\langle12\rangle^2} = \frac{\langle12\rangle^2[34]^2}{s}\frac{\cancel{(-)\langle14\rangle[43]}}{\cancel{\langle12\rangle[23]}}\,. \quad (1.356)$$

In either case, we get the same contribution, thus

$$\lim_{t\to0} F^{abcd}(s,t,u) = \frac{1}{st} f^{ade} f^{ebc}\,. \quad (1.357)$$

The u-channel intermediate state is obtained from the t-channel one, by swapping $3 \leftrightarrow 4$ and $c \to d$, so

$$\lim_{u\to0} F^{abcd}(s,t,u) = \frac{1}{su} f^{ace} f^{ebd}\,. \quad (1.358)$$

Unitarity implies that the four-point amplitude should only have single poles in each channel, but the residue in one channel always has a pole in another channel.

Since $[F] = [M]^{-4}$, we can write

$$F^{abcd}(s,t,u) = \frac{1}{st} f^{abcd}\big(\frac{s}{t}\big) + \frac{1}{ut} g^{abcd}\big(\frac{u}{t}\big)\,, \tag{1.359}$$

bearing in mind that since $s + t + u = 0$, the two ratios s/t and u/t are related.

Then we Taylor expand f and g,

$$F^{abcd}(s,t,u) = \frac{1}{st} \sum_{n=0}^{\infty} f_n^{abcd}\big(\frac{s}{t}\big)^n + \frac{1}{ut} \sum_{n=0}^{\infty} g_n^{abcd}\big(\frac{u}{t}\big)^n\,, \tag{1.360}$$

where negative values of n are not allowed, else we would get poles stronger than $1/s$ or $1/u$. The $s \to 0$ limit implies that $f_0^{abcd} = -f^{abe}f^{ecd}$. In the $u \to 0$ limit, $s = -t$, so $g_0^{abcd} = -f^{ace}f^{ebd}$.

In the $t \to 0$ limit, $u = -s$, so

$$\lim_{t\to 0} F^{abcd}(s,t,u) = \frac{1}{st} \sum_{n=0}^{\infty} \big(f_n^{abcd} - (-1)^n g_n^{abcd}\big)\big(\frac{s}{t}\big)^n = \frac{1}{st} f^{ade}f^{ebc}\,, \tag{1.361}$$

so

$$f^{ade}f^{ebc} = \sum_{n=0}^{\infty} \big(f_n^{abcd} - (-1)^n g_n^{abcd}\big)\big(\frac{s}{t}\big)^n\,. \tag{1.362}$$

Since the left-hand side is a constant, this implies that

$$f_n^{abcd} - (-1)^n g_n^{abcd} = 0, \qquad \text{for} \qquad n > 0\,, \tag{1.363}$$

$$f^{ade}f^{ebc} = f_0^{abcd} - g_0^{abcd} = -f^{abe}f^{ecd} + f^{ace}f^{ebd}\,, \tag{1.364}$$

that is, the Jacobi identity,

$$f^{abe}f^{ecd} + f^{ace}f^{edb} + f^{ade}f^{ebc} = 0\,. \tag{1.365}$$

Computing how the four-gluon amplitude factorises on all the possible two-particle channels, we have found that the only allowed colour algebra is the one of $SU(N)$, so Yang-Mills theories are the only interacting theories with massless spin-1 particles.

1.13 On-shell recursion relations

In the introduction, we said that the helicity amplitudes allow us to streamline the traditional workflow: Lagrangian → Feynman rules → Feynman diagrams → scattering amplitude → squared amplitude → cross section - by eliminating the bottleneck in squaring the amplitude.

In sec. 1.11, we took stock of that, by listing the basic processes of e^+e^- colliders and hadron colliders to which we have applied that streamlining procedure.

In this section we introduce a stark departure from that workflow, either traditional or streamlined: the on-shell recursion relations allow us to compute amplitudes without making any reference to a Lagrangian or to Feynman diagrams.

That proposes the deep question of whether it is possible to construct a quantum field theory, which is based on the fundamental pillars of quantum mechanics and special relativity, but which does not rely on off-shell structures, as quantum field theories usually do, a sort (quoting Lance Dixon) of quantum field theory without quantum fields.

1.13.1 BCFW recursion relations

The idea behind the Britto-Cachazo-Feng-Witten (BCFW) recursion relations [2] is to consider the amplitude $A_n(p_1, \ldots, p_n)$ as an analytic function of its complex momenta $p_1, \ldots, p_n$. The momenta are complexified by introducing a shift of the momenta, which preserves on-shellness and momentum conservation, and which is linear in a complex variable z. Then the amplitude $A_n(p_1, \ldots, p_n)$ becomes an analytic function of z.

Let us shift a momentum p_j^μ by a vector q^μ,

$$\hat{p}_j(z) = p_j + zq\,, \tag{1.366}$$

with $j = 1, \ldots, n$. In order to preserve total momentum, let us shift another momentum,

$$\hat{p}_i(z) = p_i - zq\,. \tag{1.367}$$

On-shellness requires that $\hat{p}_j^2(z) = \hat{p}_i^2(z) = 0$ which implies the orthogonality conditions,

$$q^2 = p_j \cdot q = p_i \cdot q = 0\,. \tag{1.368}$$

Taking p_j and p_i on the light-cone,

$$p_j = (p_j^+, p_j^-; 0_\perp)\,, \qquad p_i = (p_i^+, p_i^-; 0_\perp)\,, \tag{1.369}$$

we realise that q^μ must be a null vector in the transverse plane, $q^\mu = (0, 0; q_\perp)$. But then $q^2 = -q_\perp^2$, which cannot vanish unless q^μ is complex. A solution for q^μ is

$$q^\mu = \frac{1}{2} \left\langle j^+ \right| \gamma^\mu \left| i^+ \right\rangle = \frac{1}{2} \tilde{\lambda}_{\dot{a}}(p_j)(\sigma^\mu)^{\dot{a}a} \lambda_a(p_i)\,, \tag{1.370}$$

which fulfils the orthogonality conditions. Note that

$$(\not{q})_{b\dot{b}} = (q \cdot \overline{\sigma})_{b\dot{b}} = \frac{1}{2} \tilde{\lambda}_{\dot{a}}(p_j)(\sigma^\mu)^{\dot{a}a} \lambda_a(p_i)(\overline{\sigma}_\mu)_{b\dot{b}} = \lambda_b(p_i)\tilde{\lambda}_{\dot{b}}(p_j)\,, \tag{1.371}$$

whose short-hand is $\not{q} = \lambda_i \tilde{\lambda}_j$. Of course, another solution for p^μ is

$$q^\mu = \frac{1}{2} \left\langle i^+ \right| \gamma^\mu \left| j^+ \right\rangle \quad \Rightarrow \quad \not{q} = \lambda_j \tilde{\lambda}_i\,. \tag{1.372}$$

Let us choose $\not{q} = \lambda_i \tilde{\lambda}_j$ (1.371), so the shifts become

$$\begin{aligned} (\hat{\not{p}}_j(z))_{a\dot{a}} &= (\not{p}_j)_{a\dot{a}} + z(\not{q})_{a\dot{a}} \\ &= \lambda_a(p_j)\tilde{\lambda}_{\dot{a}}(p_j) + z\lambda_a(p_i)\tilde{\lambda}_{\dot{a}}(p_j) \\ &= (\lambda_a(p_j) + z\lambda_a(p_i))\tilde{\lambda}_{\dot{a}}(p_j)\,, \end{aligned} \tag{1.373}$$

$$\begin{aligned} (\hat{\not{p}}_i(z))_{a\dot{a}} &= \lambda_a(p_i)\tilde{\lambda}_{\dot{a}}(p_i) - z\lambda_a(p_i)\tilde{\lambda}_{\dot{a}}(p_j) \\ &= \lambda_a(p_i)(\tilde{\lambda}_{\dot{a}}(p_i) - z\tilde{\lambda}_{\dot{a}}(p_j))\,. \end{aligned} \tag{1.374}$$

In short, we can say that the shift (1.366) and (1.367), with $\not{q} = \lambda_i \tilde{\lambda}_j$ (1.371), is realised through the shift on the spinor variables,

$$\begin{cases} \hat{\lambda}_j = \lambda_j + z\lambda_i, & \hat{\tilde{\lambda}}_j = \tilde{\lambda}_j\,, \\ \hat{\tilde{\lambda}}_i = \tilde{\lambda}_i - z\tilde{\lambda}_j, & \hat{\lambda}_i = \lambda_i\,. \end{cases} \tag{1.375}$$

The amplitude $A_n(z)$ is an analytic function of the shift z above. We can use Cauchy's theorem and compute $A(z)$ over a circle large enough to encompass all its poles. If $A_n(z) \to 0$ as $z \to \infty$, then the sum of all its residues vanishes,

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\oint_C dz \frac{A(z)}{z} \qquad \text{with radius(C)} \to \infty \\ &= A_n(0) + \sum_k \text{Res}\left(\frac{A_n(z)}{z}\right)\Bigg|_{z=z_k}\,, \end{aligned} \tag{1.376}$$

where $A_n(0)$ corresponds to the original amplitude and z_k are the locations of the factorisation poles,

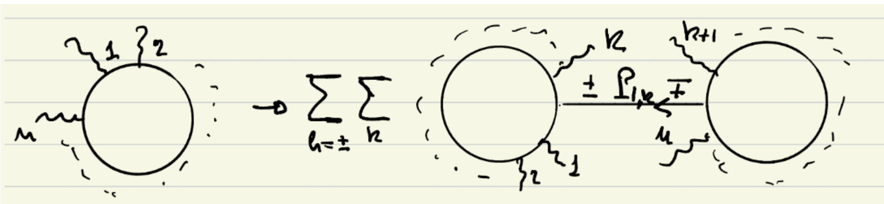


Figure 1.26: Multi-particle factorisation, with $\hat{P}^\mu_{1,k} = p^\mu_1 + \ldots + \hat{p}_j(z) + \ldots + p^\mu_k$.

Where are the poles? We said that unitarity implies that the poles of amplitudes correspond to the exchange of on-shell intermediate states. So the singularities of amplitudes stem from the vanishing propagators in the Feynman diagrams. At tree level, this means that the amplitude, or Feynman diagram, factorises into two smaller amplitudes connected by a vanishing propagator.

The momentum $\hat{P}_{1,k}$ flowing through the propagator depends on z only if particles i and j are on opposite sides of the factorising diagram. If both i and j are on the same side of the diagram, $P_{1,k}$ is independent of z and there is no pole.

We choose the shift,

$$\hat{p}_1(z) = p_1 + zq, \qquad \hat{p}_n(z) = p_n - zq, \qquad \not{q} = \lambda_n \tilde{\lambda}_1, \, . \tag{1.377}$$

i.e.

$$\begin{cases} \hat{\lambda}_1 = \lambda_1 + z\lambda_n, & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \, , \\ \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1, & \hat{\lambda}_n = \lambda_n \, . \end{cases} \tag{1.378}$$

The factorization poles in the $\sum_k$ occur when $\hat{P}_{1,k} = \hat{p}_1(z_k) + p_2 + \cdots + p_k$ is on-shell. Then,

$$\begin{aligned} 0 = \hat{P}_{1,k}^2 &= (p_1 + z_k q + p_2 + \cdots + p_k)^2 \\ &= (P_{1,k} + z_k q)^2 \\ &= P_{1,k}^2 + 2 z_k P_{1,k} \cdot q \, , \end{aligned} \tag{1.379}$$

with $P_{1,k} = p_1 + \ldots + p_k$. Thus the poles are at

$$\begin{aligned} z_k &= -\frac{P_{1,k}^2}{2 P_{1,k} \cdot q} \\ &= -\frac{P_{1,k}^2}{\langle 1^+ | \not{P}_{1,k} | n^+ \rangle} = -\frac{P_{1,k}^2}{\langle n^- | \not{P}_{1,k} | 1^- \rangle} \, , \end{aligned} \tag{1.380}$$

using eq. (1.370). If $A_n(z) \to 0$ as $z \to \infty$, Cauchy's theorem says that

$$\begin{aligned} iA_n(0) &= -\sum_k \mathrm{Res}\big(\frac{iA_n(z)}{z}\big)\bigg|_{z=z_k} \\ &= -\sum_{h=\pm} \sum_{k=2}^{n-2} \mathrm{Res}\big[iA_{k+1}(\hat{1}, 2, \ldots, k, -\hat{P}_{1,k}^{-h}) \frac{i}{z \hat{P}_{1,k}^2} iA_{n-k+1}(\hat{P}_{1,k}^h, k+1, \ldots, \hat{n})\big]\bigg|_{z=z_k} \end{aligned} \tag{1.381}$$

with A_{k+1} and A_{n-k+1} teh amplitudes on either side of the pole. Now, $z - z_k$ with z_k as in eq. (1.380), at the pole is

$$z - z_k \simeq \frac{P_{1,k}^2 + 2 z_k P_{1,k} \cdot q}{2 P_{1,k} \cdot q} \, , \tag{1.382}$$

so at the pole the propagator behaves as

$$z \hat{P}_{1,k}^2 \to z_k \, (z - z_k) \, 2 P_{1,k} \cdot q = -P_{1,k}^2 (z - z_k) \, , \tag{1.383}$$

using eq. (1.380). Thus, the amplitude is given by the recursion relation,

$$iA_n(p_1, \ldots, p_n) = \sum_{h=\pm} \sum_{k=2}^{n-2} \big[iA_{k+1}(\hat{1}, 2, \ldots, k, -\hat{P}_{1,k}^{-h}) \frac{i}{P_{1,k}^2} iA_{n-k+1}(\hat{P}_{1,k}^h, k+1, \ldots, \hat{n})\big] \, , \tag{1.384}$$

with the shifted momenta to be evaluated at $z = z_k$,

$$(\hat{\not{p}}_1(z_k))_{a\dot{a}} = \big(\lambda_1 - \frac{P_{1,k}^2}{\langle n^-|\not{P}_{1,k}|1^-\rangle}\lambda_n\big)\tilde{\lambda}_1\,,$$
$$(\hat{\not{p}}_n(z))_{a\dot{a}} = \lambda_n\big(\tilde{\lambda}_n + \frac{P_{1,k}^2}{\langle n^-|\not{P}_{1,k}|1^-\rangle}\tilde{\lambda}_1\big)\,, \tag{1.385}$$

and the sum is over the $(n-3)$ partitions of the n momenta into two sets, with at least three momenta (a three-point amplitude) on the left ($k \geq 2$) or on the right ($k \leq n-2$).

In order to complete the proof of the on-shell recursion relation (1.384), one must show that $A_n(z) \to 0$ as $z \to \infty$. Let us take gluon 1 with positive helicity, and gluon n with negative helicity. This is known as the $|-,+\rangle$ case. We consider the large z-behaviour of a generic Feynman diagram.

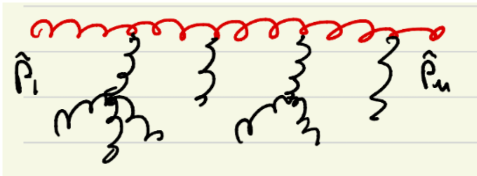


Figure 1.27: Gluon line for gluons 1 and n.

Since $\hat{p}_1(z) = p_1 + zq$, the red propagators $1/\hat{P}_{1,k}^2 = (P_{1,k}^2 + z\langle n^-|\not{P}_{1,k}|1^-\rangle)^{-1}$ grow as $1/z$ as $z \to \infty$. The 3-gluon vertices are linear in the momentum. Since there can be at most one more 3-gluon vertices than propagators, the diagram can at worst diverge as z for $z \to \infty$. Then we must include the polarisation vectors,

$$(\not{\epsilon}^+(\hat{p}_1, q))^{\dot{a}a} = (\epsilon^+\cdot\sigma)^{\dot{a}a} = \sqrt{2}\frac{\lambda^a(q)\tilde{\lambda}^{\dot{a}}(p_1)}{\lambda^b(q)\hat{\lambda}_b(p_1)} \sim \frac{1}{z}\,, \tag{1.386}$$

since $\hat{\lambda}_1 = \lambda_1 + z\lambda_n$,

$$(\not{\epsilon}^-(\hat{p}_n, q))_{a\dot{a}} = (\epsilon^-\cdot\bar{\sigma})_{a\dot{a}} = -\sqrt{2}\frac{\tilde{\lambda}_{\dot{a}}(q)\lambda_a(p_n)}{\tilde{\lambda}_{\dot{b}}(q)\hat{\tilde{\lambda}}^{\dot{b}}(p_n)} \sim \frac{1}{z}\,. \tag{1.387}$$

since $\hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$. So the amplitude falls off as $1/z$ as $z \to \infty$.

Of course, this diagrammatic argument will not work for the $|+,+\rangle$, $|-,-\rangle$ and $|+,-\rangle$ cases. As expected, the $|+,-\rangle$ case diverges as z^3 as $z \to \infty$. A clean argument for the $|+,+\rangle$, $|-,-\rangle$ cases, for which the amplitude also falls off as $1/z$ as $z \to \infty$, was given in [25] in terms of a hard particle moving in a soft background field, without making reference to Feynman diagrams.

1.13.2 MHV amplitudes

Now, we are going to use the on-shell recursion relations (1.384), with the shift (1.378) and $\not{q} = \lambda_n\tilde{\lambda}_1$, to compute the MHV amplitudes.

Firstly, we note that the spinor products,

$$\begin{aligned}\langle p\hat{1}\rangle &= \langle p1\rangle + z\langle pn\rangle\,,\\ [p\hat{n}] &= [pn] - z[p1]\,,\end{aligned} \tag{1.388}$$

with $p \neq 1, n$, are linear in the shift variable, while the spinor products $\langle p\hat{n} =\rangle\langle pn\rangle$ and $[p\hat{1}] = [p1]$ are not changed by the shift. The shift implies also that

$$\langle \hat{n}\hat{1}\rangle = \langle n1\rangle, \qquad [\hat{1}\hat{n}] = [1n]\,. \tag{1.389}$$

Further, the shift vector,

$$q^\mu = \frac{1}{2}\left\langle 1^+\right|\gamma^\mu\left|n^+\right\rangle\,, \tag{1.390}$$

which fulfils the orthogonality conditions (1.368), is proportional to the polarisation vectors,

$$\begin{aligned}\epsilon^{+\mu}(p_1,p_n) &= \frac{\langle n^-|\,\gamma^\mu\,|1^-\rangle}{\sqrt{2}\langle n1\rangle} = \frac{\langle 1^+|\,\gamma^\mu\,|n^+\rangle}{\sqrt{2}\langle n1\rangle}\,,\\ \epsilon^{-\mu}(p_n,p_1) &= -\frac{\langle 1^+|\,\gamma^\mu\,|n^+\rangle}{\sqrt{2}[1n]}\,.\end{aligned} \tag{1.391}$$

In accordance to the fact that q^μ is made of the un-shifted spinors, $\not{q} = \lambda_n\tilde{\lambda}_1$, the polarisation vectors (1.391) are not deformed by the shift,

$$\epsilon^{+\mu}(\hat{p}_1,p_n) = \epsilon^{+\mu}(p_1,p_n)\,, \qquad \epsilon^{-\mu}(\hat{p}_n,p_1) = \epsilon^{-\mu}(p_n,p_1)\,. \tag{1.392}$$

Conversely,

$$\begin{aligned}\epsilon^{-\mu}(\hat{p}_1,p_n) &= -\frac{\langle n^+|\,\gamma^\mu\left|\hat{1}^+\right\rangle}{\sqrt{2}[n\hat{1}]}\\ &= \epsilon^{-\mu}(p_1,p_n) - \frac{2zp_n^\mu}{\sqrt{2}[n1]}\,,\end{aligned} \tag{1.393}$$

where we used Gordon identity. Likewise,

$$\begin{aligned}\epsilon^{+\mu}(\hat{p}_n,p_1) &= \frac{\langle 1^-|\,\gamma^\mu\,|\hat{n}^-\rangle}{\sqrt{2}\langle 1n\rangle}\\ &= \epsilon^{+\mu}(p_n,p_1) - \frac{2zp_1^\mu}{\sqrt{2}\langle 1n\rangle}\,.\end{aligned} \tag{1.394}$$

Thus, $\epsilon^{-\mu}(\hat{p}_1,p_n)$ and $\epsilon^{+\mu}(\hat{p}_n,p_1)$ must be shifted in order to keep the orthogonality conditions in place.

Firstly, we use the on-shell recursion relations to prove Parke-Taylor formula by induction. We know that it holds for three and four gluons. We assume that it holds for $(n-1)$ gluons. Let the negative-helicity gluons be j and n. The same argument about the lack of multi-particle poles of

MHV amplitudes by counting negative helicities we discussed in sec. 1.7.3 applies here. Thus, the contributions with $3 \leq k \leq n-3$ vanish.

We are left with $k=2$ and $k=n-2$. Let us suppose that $k=n-2$. If $j=n-1$, then the A_{n-1} amplitude may have at most one negative helicity gluon, and it vanishes. So, let us take $j<n-1$. For the A_{n-1} amplitude not to vanish, h must be positive,

$$iA_{n-1}(\hat{1}^+, 2^+, \ldots, j^-, \ldots, (n-2)^+, -\hat{P}^-)\frac{i}{P_{1,n-2}^2}iA_3(\hat{P}^+, (n-1)^+, n^-)\,, \tag{1.395}$$

with

$$iA_3(\hat{P}^+, (n-1)^+, n^-) = -i\frac{[\hat{P}, (n-1)]^3}{[(n-1), \hat{n}][\hat{n}\hat{P}]}\,, \tag{1.396}$$

which can be non-vanishing only if

$$\langle \hat{P}(n-1)\rangle = \langle (n-1), \hat{n}\rangle = \langle \hat{n}\hat{P}\rangle = 0\,, \tag{1.397}$$

but we know that

$$\langle (n-1), \hat{n}\rangle = \langle (n-1), n\rangle \neq 0\,, \tag{1.398}$$

because the right-handed spinor λ_n is not shifted. Let us see how this implies that all the spinor products in $A_3(\hat{P}^+, (n-1)^+, \hat{n}^-)$ vanish. The vanishing of the propagator,

$$0 = \hat{P}_{1,n-2}^2 = \hat{P}_{n-1,n}^2 = [(n-1), \hat{n}]\langle n, n-1\rangle\,, \tag{1.399}$$

implies that $[(n-1), \hat{n}] = 0$. Likewise, using momentum conservation,

$$\begin{aligned}\left|\hat{P}^+\right\rangle[\hat{P}, (n-1)] = \hat{\not{P}}_{1,n-2}\left|(n-1)^-\right\rangle &= -(\hat{\not{p}}_n + \not{p}_{n-1})\left|(n-1)^-\right\rangle \\ &= -\left|\hat{n}^+\right\rangle[\hat{n}, (n-1)] = 0\,,\end{aligned} \tag{1.400}$$

$$\left|\hat{P}^+\right\rangle[\hat{P}, \hat{n}] = -(\hat{\not{p}}_n + \not{p}_{n-1})\left|n^-\right\rangle = -\left|(n-1)^+\right\rangle[(n-1), \hat{n}] = 0\,, \tag{1.401}$$

so all the spinor products in $A_3(\hat{P}^+, (n-1)^+, \hat{n}^-)$ vanish, making $A_3(\hat{P}^+, (n-1)^+, \hat{n}^-)$ vanish too.

We are left only with $k=2$. Let us suppose that $j>2$ (the $j=2$ case is treated in app. H.28).

For the A_3 amplitude not to vanish, h must be positive,

$$iA_3(\hat{1}^+, 2^+, -\hat{P}^-)\frac{i}{P_{1,2}^2}iA_{n-1}(\hat{P}^+, 3^+, \ldots, j^-, \ldots, (n-1)^+, \hat{n}^-)\,, \tag{1.402}$$

with

$$
\begin{aligned}
iA_3(\hat{1}^+, 2^+, -\hat{P}^-) &= -i\frac{[\hat{1}2]^3}{[2(-\hat{P})][(-\hat{P})\hat{1}]} \\
&= i\frac{[12]^3}{[2\hat{P}][\hat{P}1]}\,, \qquad (1.403)
\end{aligned}
$$

since $\hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1$, and where we analytically continued, $[k(-\hat{P})] = i[k\hat{P}]$.

Also, since $\hat{\lambda}_n = \lambda_n$, by induction,

$$
\begin{aligned}
i\,A_{n-1}(\hat{P}^+, 3^+, \ldots, j^-, \ldots, (n-1)^+, \hat{n}^-) &= i\frac{\langle j\hat{n}\rangle^4}{\langle \hat{P}3\rangle\langle 34\rangle\cdots\langle (n-1), \hat{n}\rangle\langle \hat{n}\hat{P}\rangle} \\
&= i\frac{\langle jn\rangle^4}{\langle \hat{P}3\rangle\langle 34\rangle\cdots\langle (n-1), n\rangle\langle n\hat{P}\rangle}\,, \qquad (1.404)
\end{aligned}
$$

so

$$
iA_n(1^+, \ldots, j^-, \ldots, (n-1)^+, n^-) = i\frac{\langle jn\rangle^4}{\langle \hat{P}3\rangle\langle 34\rangle\cdots\langle (n-1), n\rangle\langle n\hat{P}\rangle}\frac{i}{s_{12}}i\frac{[12]^3}{[2\hat{P}][\hat{P}1]}\,. \qquad (1.405)
$$

Then we note that

$$
\begin{aligned}
&\langle n\hat{P}\rangle[\hat{P}2] = \left\langle n^-\right|\hat{\not{P}}\left|2^-\right\rangle = \left\langle n^-\right|\not{p}_1 + \not{p}_2 + \cancelto{0}{z\lambda_n\tilde{\lambda}_1}\left|2^-\right\rangle = \langle n1\rangle[12]\,, \\
&\langle 3\hat{P}\rangle[\hat{P}1] = \left\langle 3^-\right|\hat{\not{P}}\left|1^-\right\rangle = \langle 32\rangle[21]\,, \qquad (1.406)
\end{aligned}
$$

and the amplitude becomes

$$
\begin{aligned}
& iA_n(1^+, \ldots, j^-, \ldots, (n-1)^+, n^-) \\
= \;& -i\frac{\langle jn\rangle^4\cancel{[12]^3}}{(-)\langle n1\rangle\cancel{[12]}(-\langle 12\rangle\cancel{[12]})(-\langle 23\rangle\cancel{[12]})\langle 34\rangle\cdots\langle (n-1), n\rangle} \\
& = i\frac{\langle jn\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}\,, \qquad (1.407)
\end{aligned}
$$

which proves Parke-Taylor formula.

1.13.3 NMHV six-gluon amplitudes

NMHV amplitudes, defined as the ones with three negative-helicity and $(n-3)$-positive helicity gluons, appear first in the six-gluon amplitudes, which in fact display three different NMHV helicity structures, $A_6(1^+2^+3^+4^-5^-6^-)$, $A_6(1^+2^+3^-4^+5^-6^-)$ and $A_6(1^+2^-3^+4^-5^+6^-)$, up to cyclicity and reflection symmetries. However, using the photon decoupling identity, it is possible to show that $A_6(1^+2^-3^+4^-5^+6^-)$ is related to the $(++-+--)$ structure (see app. H.29).

Let us use the on-shell recursion relation to compute $A_6(1^+2^+3^+4^-5^-6^-)$, the computation of the other two NMHV helicity structures is left to apps. H.30 and H.31.

$$\begin{aligned} & iA_6(1^+2^+3^+4^-5^-6^-) \\ &= \sum_{h=\pm}\sum_{k=2}^{4}\left[iA_{k+1}(\hat{1},2^+,\ldots,k,-P_{1,k}^{-h})\frac{i}{P_{1,k}^2}iA_{6-k+1}(\hat{P}_{1,k}^h,k+1,\ldots,\hat{6}^-)\right], \end{aligned} \tag{1.408}$$

with the shift (1.378), with $n=6$. The $k=3$ case does not contribute, since

$$A_4(\hat{1}^+,2^+,3^+,-\hat{P}_{1,3}^{-h})=0\,, \tag{1.409}$$

for either h. Further, the $k=2$ case,

$$I_2 = iA_3(\hat{1}^+,2^+,-\hat{P}_{1,2}^-)\frac{i}{P_{1,2}^2}iA_5(\hat{P}_{1,2}^+,3^+,4^-,5^-,\hat{6}^-)\,, \tag{1.410}$$

and the $k=4$ case,

$$I_4 = iA_5(\hat{1}^+,2^+,3^+,4^-,-\hat{P}_{1,4}^-)\frac{i}{P_{1,4}^2}iA_3(\hat{P}_{1,4}^+,5^-,\hat{6}^-)\,, \tag{1.411}$$

are related by parity

$$\begin{aligned} &\left|1^+\right\rangle \leftrightarrow \left|6^-\right\rangle \qquad \left|2^+\right\rangle \leftrightarrow \left|5^-\right\rangle \qquad \left|3^+\right\rangle \leftrightarrow \left|4^-\right\rangle\,, \\ &\left|4^+\right\rangle \leftrightarrow \left|3^-\right\rangle \qquad \left|5^+\right\rangle \leftrightarrow \left|2^-\right\rangle \qquad \left|6^+\right\rangle \leftrightarrow \left|1^-\right\rangle\,. \end{aligned} \tag{1.412}$$

We compute the $k=2$ case. Using eq. (1.380), the pole is at

$$z_2 = -\frac{s_{12}}{\langle 6^-|\,\not{P}_{1,2}\,|1^-\rangle} = -\frac{\langle 12\rangle}{\langle 62\rangle}\,, \tag{1.413}$$

thus

$$\hat{\not{P}}_{1,2} = \not{p}_1+\not{p}_2-\frac{\langle 12\rangle}{\langle 62\rangle}\left|6^+\right\rangle\!\left\langle 1^+\right|\,. \tag{1.414}$$

Further,

$$\left|\hat{1}^-\right\rangle = \left|1^-\right\rangle\,, \qquad \left|\hat{6}^-\right\rangle = \left|6^-\right\rangle+\frac{\langle 12\rangle}{\langle 62\rangle}\left|1^-\right\rangle\,. \tag{1.415}$$

We already computed $A_3(\hat{1}^+,2^+,-\hat{P}^-)$ in eq. (1.403), so

$$\begin{aligned} I_2 &= i\frac{[12]^3}{[2\hat{P}][\hat{P}1]}\frac{i}{s_{12}}(-i)\frac{[\hat{P}3]^3}{[34][45][5\hat{6}][\hat{6}\hat{P}]} \\ &= \frac{i}{s_{12}}\frac{[12]^3}{[2\hat{P}]\langle\hat{P}6\rangle\langle 6\hat{P}\rangle[\hat{P}1]}\frac{(\langle 6\hat{P}\rangle[\hat{P}3])^3}{[34][45][5\hat{6}][\hat{6}\hat{P}]\langle\hat{P}6\rangle}\,, \end{aligned} \tag{1.416}$$

where in the second line we multiply and divide by $\langle 6\hat{P}\rangle^3$.

Then using eqs. (1.414) and (1.415), we can make the shifted spinor products explicit

$$\langle 6\hat{P}\rangle[\hat{P}k] = \left\langle 6^-\right| (\not{p}_1 + \not{p}_2 + \cancelto{0}{z_2 \left|6^+\right\rangle\left\langle 1^+\right|}) \left|k^-\right\rangle , \qquad \forall k , \tag{1.417}$$

$$[5\hat{6}] = [56] + \frac{\langle 12\rangle}{\langle 62\rangle}[51] = \frac{\langle 2^-|\,(\not{p}_6 + \not{p}_1)\,|5^-\rangle}{\langle 62\rangle} , \tag{1.418}$$

$$\begin{aligned}
\langle 6\hat{P}\rangle[\hat{P}\hat{6}] &= \langle 6\hat{P}\rangle[\hat{P}6] + \frac{\langle 12\rangle}{\langle 62\rangle}\langle 6\hat{P}\rangle[P1] \\
&= \left\langle 6^-\right| (\not{p}_1 + \not{p}_2) \left|6^-\right\rangle + \frac{\langle 12\rangle}{\langle 62\rangle} \left\langle 6^-\right| \not{p}_2 \left|1^-\right\rangle \\
&= s_{16} + s_{26} + \frac{\langle 12\rangle}{\langle 62\rangle}\langle 62\rangle[21] \\
&= s_{612} \, .
\end{aligned} \tag{1.419}$$

Thus,

$$\begin{aligned}
I_2 &= i\frac{\cancel{[12]^3}}{\cancel{\langle 62\rangle}\cancel{[21]}\langle 61\rangle\cancel{[12]}}\frac{1}{\cancel{[21]}\langle 12\rangle}\frac{\langle 6^-|\,(\not{p}_1 + \not{p}_2)\,|3^-\rangle^3}{[34][45]s_{612}\dfrac{\left\langle 2^-\right| (\not{p}_6 + \not{p}_1) \left|5^-\right\rangle}{\cancel{\langle 62\rangle}}} \\
&= i\frac{\langle 6^-|\,(\not{p}_1 + \not{p}_2)\,|3^-\rangle^3}{\langle 61\rangle\langle 12\rangle[34][45]s_{612}\,\langle 2^-|\,(\not{p}_6 + \not{p}_1)\,|5^-\rangle} \, .
\end{aligned} \tag{1.420}$$

Using the parity symmetry, we write the $k = 4$ term,

$$I_4 = i\frac{\langle 1^+|\,(\not{p}_6 + \not{p}_5)\,|4^+\rangle^3}{[16][65]\langle 43\rangle\langle 32\rangle s_{561}\,\langle 5^+|\,(\not{p}_1 + \not{p}_6)\,|2^+\rangle} , \tag{1.421}$$

and finally

$$\begin{aligned}
iA_6(1^+2^+3^+4^-5^-6^-) &= i\frac{\langle 6^-|\,(\not{p}_1 + \not{p}_2)\,|3^-\rangle^3}{\langle 61\rangle\langle 12\rangle[34][45]s_{612}\,\langle 2^-|\,(\not{p}_6 + \not{p}_1)\,|5^-\rangle} \\
&+ i\frac{\langle 4^-|\,(\not{p}_5 + \not{p}_6)\,|1^-\rangle^3}{\langle 23\rangle\langle 34\rangle[56][61]s_{561}\,\langle 2^-|\,(\not{p}_6 + \not{p}_1)\,|5^-\rangle} \, .
\end{aligned} \tag{1.422}$$

The $A_6(1^+2^+3^+4^-5^-6^-)$ amplitude had been derived already by Mangano, Parke and Xu [27]. The advantage of the expression above is that it is shorter and with a simpler singularity structure, although it contains a spurious singularity, given by $\langle 2^-|\,(\not{p}_6 + \not{p}_1)\,|5^-\rangle \to 0$, which may occur when $p_6{+}p_1$ is a linear combination of p_2 and p_5. The singularity is spurious because although the amplitude is finite in that kinematic point, individual terms of the amplitude are singular (see the discussion in [9]).

1.14 Gravitation

Let us consider general relativity and the Einstein-Hilbert action, without matter fields,

$$S_{EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R\,, \tag{1.423}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, with $R_{\mu\nu}$ the Ricci curvature tensor, and $k^2 = 8\pi G_N$, with Newton's constant G_N. The variation $\delta g_{\mu\nu}$ of the gravitational field $g_{\mu\nu}$ yields the gravity part of Einstein's equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\,. \tag{1.424}$$

If we expand $g_{\mu\nu}$ around the flat space $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{1.425}$$

we may consider $h_{\mu\nu}$ as the graviton field. Since $R_{\mu\nu}$ involves two derivatives of the field $g_{\mu\nu}$, all terms in the expansion of the action will display two derivatives of $h_{\mu\nu}$, i.e. schematically,

$$S_{EH} = \frac{1}{2k^2}\int d^4x[h\partial^2h + \kappa h^2\partial^2h + \kappa^2h^3\partial^2h + \ldots]\,. \tag{1.426}$$

After gauge fixing (e.g through the De Donder gauge), the $h\partial^2h$ term will yield the graviton propagator, $h^2\partial^2h$ the three-graviton vertex, $h^3\partial^2h$ the four-graviton vertex, and in general $h^{n-1}\partial^2h$ will yield the n-graviton vertex. The graviton field $h_{\mu\nu}$ has spin 2, and its polarisation tensor is given simply by the product of two spin-1 polarisation vectors,

$$\epsilon^{\mu\nu}_{\pm}(p) = \epsilon^{\mu}_{\pm}(p)\epsilon^{\nu}_{\pm}(p)\,. \tag{1.427}$$

The graviton is massless, so it has two helicity states.

Computing amplitudes out of the Feynman rules for the graviton vertices is very complicated, even for the simplest, i.e. the four-graviton tree amplitude. Yet, at fixed helicities the outcome is very simple,

$$M_4^{tree}(1^-2^-3^+4^+) = \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}\,. \tag{1.428}$$

In the spirit of the on-shell recursion relations, we will bypass Einstein-Hilbert action, the Feynman rules for the graviton vertices, and compute graviton amplitudes directly on-shell.

1.14.1 Three-graviton amplitudes

In sec. 1.12.3, we have seen that for a theory of massless particles of a given integer spin s, little group scaling and dimensional analysis arguments fix uniquely the dependence of the three-boson amplitudes on the spinor products,

$$
\begin{aligned}
M_3^{tree}(1^-, 2^-, 3^+) &\propto \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^s , \\
M_3^{tree}(1^+, 2^+, 3^-) &\propto \left(\frac{[12]^3}{[23][31]}\right)^s .
\end{aligned}
$$

Gravitons have helicity $= \pm 2$, so the same arguments fix the three-graviton amplitudes to be,

$$
\begin{aligned}
M_3^{tree}(1^-, 2^-, 3^+) &\propto \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2} , \\
M_3^{tree}(1^+, 2^+, 3^-) &\propto \frac{[12]^6}{[13]^2[23]^2} .
\end{aligned}
\tag{1.429}
$$

In particular, we fix the three-graviton amplitudes to be the square of the three-gluon amplitudes (we will see later why),

$$
\begin{aligned}
iM_3^{tree}(1^-, 2^-, 3^+) &= i(iA_3^{tree}(1^-, 2^-, 3^+))^2 , \\
iM_3^{tree}(1^+, 2^+, 3^-) &= i(iA_3^{tree}(1^+, 2^+, 3^-))^2 .
\end{aligned}
\tag{1.430}
$$

1.14.2 Uniqueness of General Relativity

Examining the three-particle amplitudes in sec. 1.12.3, we have already seen that Bose symmetry severely constrains theories of self-interacting massless particles of integer spin s: they must have an even spin, odd spins being only allowed with at least three particles and a fully antisymmetric coupling. Now, we want to use four-particle amplitudes to show that the graviton, and so linearised General Relativity, is the only allowed self-interacting massless particle of (even) integer spin s.

Using the on-shell recursion relations, a four-particle amplitude can be constructed through at most two of the three channels. For example, let us suppose that the shift involves particles 1 and 4. They must be on opposite sides of the on-shell propagator. That excludes then the s_{14} channel.

Of course, for self-consistency, the four-particle amplitude computed in this way must not depend on the shift. We shall use it as a selection criterion for possible theories. Let us consider a self-interacting massless particle of integer spin s, whose three-particle amplitudes are given by (stripped off of couplings)

$$
\begin{aligned}
M_3(1^-, 2^-, 3^+) &= i^{s-1}\left(i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^s , \qquad M_3(1^-, 2^-, 3^-) = 0 , \\
M_3(1^+, 2^+, 3^-) &= i^{s-1}\left(-i\frac{[12]^3}{[23][31]}\right)^s , \qquad M_3(1^+, 2^+, 3^+) = 0, .
\end{aligned}
\tag{1.431}
$$

In the Tutorial, we compute the amplitude $M_4(1^+,2^-,3^+,4^-)$ using the $|4^-,1^+\rangle$ shift,

$$\begin{cases} \hat{\lambda}_1 = \lambda_1 + z\lambda_4, & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1\,, \\ \hat{\tilde{\lambda}}_4 = \tilde{\lambda}_4 - z\tilde{\lambda}_1, & \hat{\lambda}_4 = \lambda_4\,. \end{cases}, \tag{1.432}$$

with $\not{q} = \lambda_4\tilde{\lambda}_1$. The amplitude is un-ordered, so it has two contributions, given in fig. 1.28, which yield (see app. H.32)

$$iM_4^{(4,1)}(1^+,2^-,3^+,4^-) = i\,\left([13]^2\langle 24\rangle^2\right)^s \frac{(s_{14})^{2-s}}{s_{12}s_{13}s_{14}}\,, \tag{1.433}$$

which has the correct little group scaling.

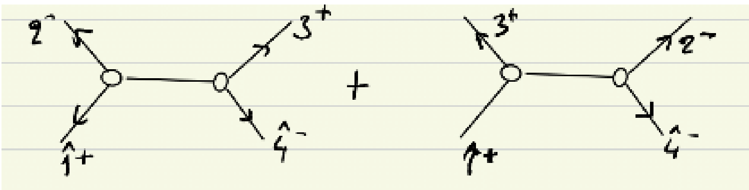


Figure 1.28: Amplitude $M_4(1^+,2^-,3^+,4^-)$ under $|4^-,1^+\rangle$ shift.

We compute again the amplitude $M_4(1^+,2^-,3^+,4^-)$ through the $|2^-,1^+\rangle$ shift,

$$\begin{cases} \hat{\lambda}_1 = \lambda_1 + z\lambda_2, & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1\,, \\ \hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 - z\tilde{\lambda}_1, & \hat{\lambda}_2 = \lambda_2\,. \end{cases} \quad \text{with} \quad \not{q} = \lambda_2\tilde{\lambda}_1\,. \tag{1.434}$$

In order to do that, we can use the fact that by reflection,

$$M_4^{(2,1)}(1^+,2^-,3^+,4^-) = M_4^{(2,1)}(1^+,4^-,3^+,2^-)\,. \tag{1.435}$$

The amplitude on the right-hand side can be obtained from $M_4^{(4,1)}(1^+,2^-,3^+,4^-)$ by swapping labels 2 and 4 in eq. (1.433), so that

$$iM_4^{(2,1)}(1^+,2^-,3^+,4^-) = i\,\left([13]^2\langle 24\rangle^2\right)^s \frac{(s_{12})^{2-s}}{s_{12}s_{13}s_{14}}\,. \tag{1.436}$$

Since the computation of the four-particle amplitude must not depend on the shift,

$$M_4^{(2,1)}(1^+,2^-,3^+,4^-) = M_4^{(4,1)}(1^+,2^-,3^+,4^-)\,, \tag{1.437}$$

we obtain that $(s_{14})^{2-s} = (s_{12})^{2-s}$. For this to be true, for every s_{12} and s_{14}, we must have $s = 2$ [29].

Note that for $s = 2$, the amplitude has the correct residues in all three channels.

Further, note that for $(1^-,2^-,3^+,4^+)$ we obtain

$$M_4(1^-,2^-,3^+,4^+) = \frac{\langle 12\rangle^4[34]^4}{s_{12}s_{13}s_{14}}\,, \tag{1.438}$$

in agreement with eq. (1.428) (please check it).

1.14.3 Multi-graviton MHV amplitudes

Just like for gluons, supersymmetric Ward identities allow one to prove that the graviton amplitudes with all-like helicity gravitons, or all but one, vanish at tree level,

$$\begin{aligned} M_n^{tree}(1^\pm, 2^\pm, \ldots, n^\pm) &= 0\,, \\ M_n^{tree}(1^\mp, 2^\pm, \ldots, n^\pm) &= 0\,. \end{aligned} \tag{1.439}$$

Just like for Yang-Mills, tree graviton amplitudes cannot tell if they belong to a pure gravity theory or a supersymmetric extension (supergravity).

Just like gluon amplitudes, graviton amplitudes are classified as MHV, NMHV, and so on. Tree MHV graviton amplitudes can be obtained though on-shell recursion relations, and expressed as the square of tree MHV gluon amplitudes, as we will see.

In the Tutorial, we consider the MHV amplitude $M_n^{tree}(1^-, 2^-, 3^+, \ldots, n^+)$, and by induction we show that it takes the form [30],

$$iM_n^{tree}(1^-, 2^-, 3^+, \ldots, n^+) = i \sum_{\sigma \in S_{n-2}} 2p_1 \cdot p_{\sigma_n} \Big(\prod_{k=4}^{n-1} \beta_k \Big) \big(iA_n(1^-, 2^-, \sigma_3^+, \ldots, \sigma_n^+) \big)^2\,, \tag{1.440}$$

$$\text{with} \qquad \beta_k = \begin{cases} -\dfrac{\langle \sigma_k \sigma_{k+1} \rangle}{\langle 2 \sigma_{k+1} \rangle} \left\langle 2^- \middle| \not{P}_{\sigma_3, \sigma_{k-1}} \middle| \sigma_k^- \right\rangle & \text{for} \qquad n > 4 \\ 1, & \text{for} \qquad n = 4\,. \end{cases} \tag{1.441}$$

and $P_{i,j} = p_i + \ldots + p_j$. In particular, for $n = 4$ eq. (1.440) yields

$$iM_4^{tree}(1^-, 2^-, 3^+, 4^+) = i \big[s_{14} (iA_4(1^-, 2^-, 3^+, 4^+))^2 + s_{13} (iA_4(1^-, 2^-, 4^+, 3^+))^2 \big]\,, \tag{1.442}$$

which can be shown (see Tutorial) to agree with the usual form of the four-graviton amplitude (1.428).

Eq. (1.440) has manifest S_{n-2} permutation symmetry over $(n-2)$ gravitons, but of course the amplitude is fully symmetric over n gravitons, so we introduce a formula [31] for the n-graviton MHV amplitude, which is manifestly symmetric over n gravitons.

Firstly, we introduce the symmetric function,

$$\begin{cases} \varphi^i{}_j = \dfrac{[ij]}{\langle ij \rangle} & j \neq i\,, \\ \varphi^i{}_i = -\displaystyle\sum_{k \neq i} \dfrac{[ik] \langle kx \rangle \langle ky \rangle}{\langle ik \rangle \langle ix \rangle \langle iy \rangle}\,. \end{cases} \tag{1.443}$$

The function $\varphi^i{}_i$ does not depend on the spinors x, y. To see it, let us change x with x',

$$\sum_{k\neq i} \frac{[ik]\langle kx'\rangle\langle ky\rangle}{\langle ik\rangle\langle ix'\rangle\langle iy\rangle} = \sum_{k\neq i} \frac{[ik]\langle kx\rangle\langle ky\rangle}{\langle ik\rangle\langle ix\rangle\langle iy\rangle} \frac{\langle kx'\rangle\langle ix\rangle}{\langle ix'\rangle\langle kx\rangle} . \tag{1.444}$$

By Schouten identity,

$$\langle ix\rangle\langle kx'\rangle + \langle ik\rangle\langle x'x\rangle + \langle ix'\rangle\langle xk\rangle = 0 , \tag{1.445}$$

thus we get

$$\sum_{k\neq i} \frac{[ik]\langle kx\rangle\langle ky\rangle}{\langle ik\rangle\langle ix\rangle\langle iy\rangle}\left(1 + \frac{\langle ik\rangle\langle xx'\rangle}{\langle ix'\rangle\langle kx\rangle}\right) = -\varphi^i{}_i + \sum_{k\neq i} \frac{[ik]\langle ky\rangle\langle xx'\rangle}{\langle ix\rangle\langle iy\rangle\langle ix'\rangle} , \tag{1.446}$$

and the second term vanishes by momentum conservation.

The functions $\varphi^i{}_j$, form an $n\times n$ symmetric matrix ϕ, out of which we construct the $(n-3)\times(n-3)$ minor determinant $|\phi|^{ijk}_{pqr}$ obtained by deleting rows i, j, k and columns p, q, r. Further, we introduce the coefficient,

$$c^{ijk} = c_{ijk} = \frac{1}{\langle ij\rangle\langle jk\rangle\langle ki\rangle} . \tag{1.447}$$

Then we write the amplitude M_n as

$$M_n(1, 2, \ldots, n) = \langle ij\rangle^8 \widetilde{M}_n(1, 2, \ldots, n) , \tag{1.448}$$

where i and j are the negative-helicity gravitons and $\widetilde{M}_n$ is helicity independent. $\widetilde{M}_n$ can be written as [31]

$$\widetilde{M}_n(1, 2, \ldots, n) = (-1)^{n+1}\mathrm{sgn}(ijk)\ \mathrm{sgn}(rst) c_{ijk} c^{rst} |\phi|^{ijk}_{rst} , \tag{1.449}$$

where $\mathrm{sgn}(ijk) \equiv \mathrm{sgn}(\sigma(i, j, k, 1, 2, \ldots, \not{i}, \not{j}, \not{k}, \ldots, n))$ is the signature of the permutation which moves i, j, k up front in the sequence.

In order to check that M_n is symmetric under S_n permutations, it is enough to show that e.g.

$$c_{ijk}|\phi|^{ijk}_{rst} = -c_{ij\ell}|\phi|^{ij\ell}_{rst} , \tag{1.450}$$

for any $\ell \neq k$. For this, it is convenient to introduce the function,

$$f^i{}_j = \langle i1\rangle\langle i2\rangle\varphi^i{}_j . \tag{1.451}$$

Then

$$\sum_{i=1}^{n} f^i{}_j = \sum_{i\neq j} \langle i1\rangle\langle i2\rangle \frac{[ij]}{\langle ij\rangle} - \langle j1\rangle\langle j2\rangle \sum_{k\neq j} \frac{[jk]\langle kx\rangle\langle ky\rangle}{\langle jk\rangle\langle jx\rangle\langle jy\rangle} = 0 , \tag{1.452}$$

which is straightforward if we choose $x = 1, y = 2$. Thus, for a given column j, the rows i in the $n\times n$

matrix f all sum to zero. By the properties of determinants, this implies, e.g. that $|f|^{ijk}_{rst} = -|f|^{ij\ell}_{rst}$ and so that $c_{ijk}|\phi|^{ijk}_{rst} = -c_{ij\ell}|\phi|^{ij\ell}_{rst}$.

Let us check that $\widetilde{M}_n$ reproduces the known expressions for three (1.430) and four gravitons (1.428).

$$\begin{aligned} M_3(1^-,2^-,3^+) &= \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,, \\ M_4(1^-2^-3^+4^+) &= \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}\,, \end{aligned}$$

imply that

$$\widetilde{M}_3(1,2,3) = \frac{1}{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}\,, \tag{1.453}$$

$$\widetilde{M}_4(1,2,3,4) = \frac{[12]}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}\,. \tag{1.454}$$

For $n=3$, $|\phi|^{123}_{123} = 1$, and we get eq. (1.453).
For $n=4$, let us choose the minor $|\phi|^{234}_{234} = \varphi^1{}_1$, with $x=3, y=4$,

$$\begin{aligned} \varphi^1{}_1 &= -\frac{[12]\langle 23\rangle\langle 24\rangle}{\langle 12\rangle\langle 13\rangle\langle 14\rangle}\,, \\ \widetilde{M}_4 &= -c_{234}c^{234}\varphi^1{}_1 \\ &= \frac{1}{\langle 23\rangle^{\cancel{2}}\langle 34\rangle^2\langle 24\rangle^{\cancel{2}}}\frac{[12]\cancel{\langle 23\rangle}\cancel{\langle 24\rangle}}{\langle 12\rangle\langle 13\rangle\langle 14\rangle}\,, \end{aligned} \tag{1.455}$$

in agreement with eq. (1.454). We can check that the result does not depend on the choice of the minor, by choosing $|\phi|^{123}_{123} = \varphi^4{}_4$ with $x=2, y=3$,

$$\begin{aligned} \varphi^4{}_4 &= -\frac{[41]\langle 12\rangle\langle 13\rangle}{\langle 41\rangle\langle 42\rangle\langle 43\rangle}\,, \\ \widetilde{M}_4 &= -c_{123}c^{123}\varphi^4{}_4 \\ &= \frac{1}{\langle 12\rangle^{\cancel{2}}\langle 13\rangle^{\cancel{2}}\langle 23\rangle^2}\frac{[41]\cancel{\langle 12\rangle}\cancel{\langle 13\rangle}}{\langle 41\rangle\langle 42\rangle\langle 43\rangle}\,, \end{aligned} \tag{1.456}$$

and using momentum conservation, $[41]\langle 43\rangle = [12]\langle 23\rangle$ we get again eq. (1.454). If, say, we choose the minor $|\phi|^{234}_{123} = \varphi^1{}_4$,

$$\begin{aligned} \varphi^1{}_4 &= \frac{[14]}{\langle 14\rangle}\,, \\ \widetilde{M}_4 &= c_{234}c^{123}\varphi^1{}_4 \\ &= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\langle 23\rangle\langle 34\rangle\langle 42\rangle}\frac{[14]}{\langle 14\rangle}\,, \end{aligned} \tag{1.457}$$

and using momentum conservation we get yet again eq. (1.454).

1.14.4 Kawai-Lewellen-Tye relations

Let us write the four-graviton amplitude as in eq. (1.442),

$$iM_4(1^-,2^-,3^+,4^+) = i\big[s_{14}(iA_4(1^-,2^-,3^+,4^+))^2 + s_{13}(iA_4(1^-,2^-,4^+,3^+))^2\big]\,.$$

Using reflection and the photon decoupling with respect to gluon 2, we write

$$A_4(1,2,4,3) = A_4(1,3,4,2) = -A_4(1,2,3,4) - A_4(1,3,2,4)\,. \tag{1.458}$$

Then we use the BCJ relation (1.235), and we write eq. (1.458) as

$$\begin{aligned} A_4(1,2,4,3) &= -\left(1+\frac{s_{12}}{s_{13}}\right) A_4(1,2,3,4) \\ &= \frac{s_{14}}{s_{13}} A_4(1,2,3,4)\,. \end{aligned} \tag{1.459}$$

Thus we can write eq. (1.442) as

$$\begin{aligned} iM_4(1^-,2^-,3^+,4^+) &= i\big(\cancel{s_{14}}\frac{s_{13}}{\cancel{s_{14}}} + \cancel{s_{13}}\frac{s_{14}}{\cancel{s_{13}}}\big) iA_4(1^-,2^-,3^+,4^+)\; iA_4(1^-,2^-,4^+,3^+) \\ &= -is_{12} iA_4(1^-,2^-,3^+,4^+)\; iA_4(1^-,2^-,4^+,3^+)\,. \end{aligned} \tag{1.460}$$

This is the simplest example of Kawai-Lewellen-Tye (KLT) relations. They were derived in string theory, giving the n-point closed string amplitudes as a sum over products of pairs of n-point open string amplitudes. In the low-energy limit, the n-point closed string amplitudes become tree graviton amplitudes M_n, while the n-point open string amplitudes become colour-ordered gluon amplitudes A_n. Note however that the KLT relations are valid without specifying the helicity states, they are in fact valid in d spacetime dimensions.

1.14.5 Colour-kinematics duality

In sec. 1.7.4, in dealing with the four-gluon amplitude (1.211) which led to the BCJ relation, we saw that the Jacobi identity, $c_s + c_t + c_u = 0$ has a kinematic analog, $n_s + n_t + n_u = 0$, eq. (1.223).

Writing the tree amplitude as a sum over all distinct cubic diagrams (where we have eliminated the four-gluon vertices as in sec. 1.7.4),

$$A_n^{tree} = \sum_i \frac{c_i n_i}{D_i}\,, \tag{1.461}$$

BCJ proposed that every time there is a triplet of colour factors $\{c_i, c_j, c_k\}$ which are linked, $c_i + c_j + c_k = 0$, the amplitude features colour-kinematics (CK) duality if the kinematics factors are linked likewise, $n_i + n_j + n_k = 0$. In sec. 1.7.4, we have shown that four-gluon amplitudes feature CK duality, and that is true in general for Yang-Mills (i.e. pure gluon) amplitudes.

The gauge-dependent kinematic factors n_i are of course not unique. Any shift of the polarisation vectors $\epsilon(p_i) \to \epsilon(p_i) + a_i p_i$ changes the n_i's without changing the gauge-invariant amplitude. Examining again the four-gluon amplitude (1.211),

$$iM_4 = -ig^2\big(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\big)\,,$$

the non-local shifts,

$$n_s \to n_s + s\Delta\,, \qquad n_t \to n_t + t\Delta\,, \qquad n_u \to n_u + u\Delta\,, \tag{1.462}$$

which are like adding a contact term to each channel, will also leave the amplitude invariant, since $c_s + c_t + c_u = 0$.

For the n-point amplitude (1.461), if we have a set of numerators which fulfil the CK duality, $c_i + c_j + c_k = 0 \Leftrightarrow n_i + n_j + n_k = 0$, and we shift the numerators, $n_i \to n_i + \Delta_i$, with the constraint,

$$\sum_i \frac{c_i \Delta_i}{D_i} = 0\,, \tag{1.463}$$

the amplitude is invariant. The Δ_i's are like gauge functions, since they drop out of the amplitude.

Once we have the numerators fulfilling the CK duality, let us see what happens on the four-gluon amplitude (1.211) with the formal replacement,

$$c_i \to n_i\,. \tag{1.464}$$

Stripping-off the coupling constant, we get

$$iM_4(1,2,3,4) = -i\left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}\right)\,, \tag{1.465}$$

with $n_t = -(n_s + n_u)$. Then we use the relation (1.233) between n_s, n_u and $A_4(1,2,3,4)$,

$$A_4(1,2,3,4) = \frac{u}{st} n_s - \frac{1}{t} n_u\,, \tag{1.466}$$

so that

$$\begin{aligned} n_u &= \frac{u}{s}\, n_s - t A_4(1,2,3,4)\,, \\ n_t &= -\,(n_s + n_u) \\ &= -\left(1 + \frac{u}{s}\right) n_s + t A_4(1,2,3,4) \\ &= t\left(A_4(1,2,3,4) + \frac{n_s}{s}\right)\,, \end{aligned} \tag{1.467}$$

and we re-write the amplitude as

$$\begin{aligned}
iM_4(1,2,3,4) &= -i\Big[\frac{n_s^2}{s} + t\left(A_4(1,2,3,4) + \frac{n_s}{s}\right)^2 + \left(\frac{u}{s}n_s - tA_4(1,2,3,4)\right)^2 \frac{1}{u}\Big] \\
&= -i\Big[\cancel{\frac{n_s^2}{s}} + tA_4^2 + \cancel{\frac{2t}{s}n_s A_4} + \cancel{\frac{t}{s^2}n_s^2} + \frac{t^2A_4^2}{u} + \cancel{\frac{u}{s^2}n_s^2} - \cancel{\frac{2t}{s}n_s A_4}\Big] \\
&= -it\left(1 + \frac{t}{u}\right) A_4(1234)^2 \\
&= i\frac{st}{u}A_4(1234)^2\,, \qquad (1.468)
\end{aligned}$$

and using the BCJ relation (1.459), we can write

$$iM_4(1,2,3,4) = -is_{12}\; iA_4(1,2,3,4)\; iA_4(1,2,4,3)\,, \qquad (1.469)$$

i.e. the KLT relation (1.460).

That is, using the CK duality, we have found the relation between the four-graviton amplitude M_4 on the left-hand side and the colour-ordered gluon amplitude A_4 on the right-hand side. As we said for the KLT relation (1.460), eq. (1.469) does not depend on the helicities or on the dimension of space-time.

The procedure of obtaining

$$M_n^{tree} = \sum_i \frac{n_i^2}{D_i}\,, \qquad (1.470)$$

through the substitution $c_i \to n_i$ is called double copy. Thus, at least at tree level, we can say (in a sense that we will make more precise later) that

$$\text{Gravity} \subset (\text{Yang} - \text{Mills})^2\,. \qquad (1.471)$$

Just like the KLT relation (1.460), the double copy carries on at higher points. However, the KLT relations work only at the tree level, while the CK duality and the double copy are also valid at loop level.

Further, the double copy works also with different sets of Yang-Mills numerators, n_i and $\tilde{n}_i$. Suppose that the n_i's fulfil the CK duality, while the $\tilde{n}_i$'s do not manifestly. Since n_i and $\tilde{n}_i$ are valid representations of the same Yang-Mills amplitude, we can write

$$\tilde{n}_i = n_i + \Delta_i\,, \qquad (1.472)$$

with the constraint (1.463), such that the amplitude stays invariant. Since the n_i's fulfil CK duality, they can replace the colour factor in the constraint (1.463),

$$\sum_i \frac{n_i \Delta_i}{D_i} = 0\,. \qquad (1.473)$$

Then the relation

$$M_n^{tree} = \sum_i \frac{n_i \tilde{n}_i}{D_i}\,, \tag{1.474}$$

is equivalent to eq. (1.470).

In addition to combining two sets of numerators when only one fulfils manifestly the CK duality, eq. (1.474) may be very convenient also to combine numerators which refer to different external states, which through the double copy allows one to obtain states different from the graviton, and finally to combine numerators from e.g. a supersymmetric version of Yang-Mills theory with the ones of a not-necessarily supersymmetric version of it, allowing one to explore different supersymmetric extensions of gravity.

This is an active field of research, and many theories have been shown to exhibit CK duality and admit a double copy. For more details, we refer to the review on CK duality [12].

1.15 Scattering equations

We consider particles with complex momenta in a D-dimensional Minkowski space $\mathbb{C}M$, and introduce the configuration space ϕ_n of n scattering gluons,

$$\phi_n = \{(p_1, \ldots, p_n) \in (\mathbb{C}M)^n / \sum_{i=1}^n p_i = 0,\ p_1^2 = \ldots = p_n^2 = 0\} \tag{1.475}$$

i.e. all the n-tuples $p = (p_1, \ldots, p_n)$ of on-shell light-like momenta constrained by momentum conservation. The gluons are also characterised by an n-tuple $\epsilon = (\epsilon_1^{\lambda_1}, \ldots, \epsilon_n^{\lambda_n})$ of polarisation vectors fixed by the gluon helicities $\lambda_I = \pm 1$, with $i = 1, \ldots, n$. Finally, a colour-ordered n-gluon tree amplitude in D dimensions is fixed by a specific permutation $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n/\mathbb{Z}_n$ of the gluons, up to a cyclic order, so we can label it as

$$A_n(p, \epsilon, \sigma) = A_n(p_{\sigma_1}^{\lambda_{\sigma_1}}, \ldots, p_{\sigma_n}^{\lambda_{\sigma_n}})\,. \tag{1.476}$$

We consider $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$, i.e. the complex space plus the point at infinity, which can be modelled by the Riemann sphere, which is equivalent to the complex projective line $\mathbb{CP}^1$, i.e. the projective space of lines in $\mathbb{C}^2$. Since we will need it also later, we explain very basic notions of projective geometry in App. B.

We consider then the n-tuples $(z_1, \ldots, z_n) \in \hat{\mathbb{C}}^n$. Given the functions,

$$f_i(z, p) = \sum_{j \neq i} \frac{2 p_i \cdot p_j}{z_i - z_j}\,, \tag{1.477}$$

the scattering equations are [34, 35]

$$f_i(z, p) = 0 \qquad i = 1, \ldots, n\,, \tag{1.478}$$

i.e. for a given $(p_1, \ldots, p_n)$ in ϕ_n, a solution is an n-tuple $(z_1, \ldots, z_n)$ such that eqs. (1.478) are fulfilled.

Let us introduce the projective special linear group $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\mathbb{1}, -\mathbb{1}\}$, with elements

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det(g) = 1\,. \tag{1.479}$$

They are the Möbius transformations,

$$g(z) = \frac{az+b}{cz+d}, \qquad \text{with} \qquad z \in \hat{\mathbb{C}}\,. \tag{1.480}$$

If $(z_1, \ldots, z_n)$ is a solution of the scattering equations, so is $(g(z_1), \ldots, g(z_n))$ (see app. H.44), thus the scattering equations are invariant under $\mathrm{PSL}(2,\mathbb{C})$ maps.

We are only interested in the different equivalence classes $[z_1 : \ldots : z_n]$ of the $\mathrm{PSL}(2,\mathbb{C})$-invariant solutions. We define then the moduli space $M_{0,n}$ of genus zero curves with n punctures,

$$M_{0,n} = \{(z_1,\ \ldots,\ z_n) \in (\mathbb{C}P^1)^n /\ z_i \neq z_j\}/\ \mathrm{PSL}(2,\mathbb{C})\,. \tag{1.481}$$

Any inequivalent solution of the scattering equations corresponds to a point in $M_{0,n}$. Since $\mathrm{PSL}(2,\mathbb{C})$, can fix three points at $0, 1$ and ∞, $M_{0,n}$ has dimension $(n-3)$.

Through the Möbius invariance of the scattering equations, one can see that there are only $(n-3)$ independent equations. Let us consider the $\mathrm{PSL}(2,\mathbb{C})$-invariant function,

$$U(z,p) = \prod_{j<k} (z_j - z_k)^{2p_j \cdot p_k}\,, \tag{1.482}$$

with

$$U^{-1} \frac{\partial}{\partial z_i} U = f_i(z,p)\,, \tag{1.483}$$

(see app. H.45). Since an infinitesimal Möbius transformation can be written as $\delta z = \epsilon_0 + \epsilon_1 z + \epsilon_2 z^2$, with infinitesimal parameters $\epsilon_0, \epsilon_1, \epsilon_2$ (see app. H.46), then

$$\begin{aligned} 0 = \delta U &= \sum_{i=1}^{n} \frac{\partial U}{\partial z_i} \delta z_i \\ &= U \sum_{i=1}^{n} (\epsilon_0 + \epsilon_1 z_i + \epsilon_2 z_i^2) f_i(z,p)\,, \end{aligned} \tag{1.484}$$

since this must hold for every $\epsilon_0, \epsilon_1, \epsilon_2$, it implies that

$$\sum_{i=1}^{n} z_i^m f_i(z,p) = 0, \qquad m = 0, 1, 2, \tag{1.485}$$

so there are three linear relations (explicitly checked in the Tutorials) among the n functions $f_i(z,p)$, and thus $(n-3)$ independent equations.

1.15.1 The polynomial form

The scattering equations can also be put in polynomial form [36]. Let us consider the function,

$$g_m(z,p) = \sum_{i=1}^{n} z_i^m f_i(z,p)\,, \tag{1.486}$$

which vanishes for $m = 0, 1, 2.$ from eq. (1.485). Making g_m explicit, and using the antisymmetry of the denominator,

$$g_m(z,p) = \frac{1}{2}\sum_i \sum_{j\neq i} (z_i^m - z_j^m)\,\frac{2p_i\cdot p_j}{z_i - z_j}\,. \tag{1.487}$$

Then we write the power difference as

$$z_i^m - z_j^m = (z_i - z_j)\sum_{k=0}^{m-1} z_i^k z_j^{m-1-k}\,, \tag{1.488}$$

so that

$$g_m(z,p) = \frac{1}{2}\sum_i \sum_{j\neq i}\; 2p_i\cdot p_j \sum_{k=0}^{m-1} z_i^k z_j^{m-1-k}\,. \tag{1.489}$$

The $n\times n$ matrix $Z_{mi} = {z_i}^m$, with $0 \leq m \leq n-1$, $1 \leq i \leq n$ is non-singular since

$$\det Z = \prod_{1\leq i<j\leq n} (z_j - z_i)\,, \tag{1.490}$$

is the Vandermonde determinant. So the $(n-3)$ equations $g_m(z,p)$ with $3 \leq m \leq n-1$ are equivalent to the $(n-3)$ independent equations $f_i(z,p)$.

However, a more convenient form of the equations $g_m(z,p)$ is obtained by considering the set $A = \{1, 2, \ldots, n\}$ and any subset $S \subseteq A$, with

$$p_S = \sum_{i\in S} p_i\,, \qquad z_S = \prod_{i\in S} z_i\,. \tag{1.491}$$

Then for $1 \leq m \leq n$ we define the polynomials

$$h_m(z,p) = \sum_{s\subset A, |s|=m} p_S^2\; z_S\,, \tag{1.492}$$

where $|s| = m$ is the number of elements in S and the sum runs over the $\binom{n}{m} = \dfrac{n!}{m!\,(n-m)!}$

subsets S of A. For $m = 1$, we have n subsets of one element,

$$\begin{aligned} &S = \{i\}, \qquad i = 1, \dots, n\,, \\ &h_1(z,p) = \sum_{i=1}^{n} \cancelto{0}{p_i^2} z_i = 0 \end{aligned} \tag{1.493}$$

where we used on-shellness. For $m = n - 1$, we have n subsets of $(n-1)$ elements,

$$\begin{aligned} &S = \{1, \dots, i-1, i+1, \dots, n\}, \quad i = 1, \dots, n\,, \quad p_S = \sum_{j \neq i}^{n} p_j = -p_i, \quad z_S = \prod_{j\neq i} z_j\,, \\ &h_{n-1}(z,p) = \sum_{i=1}^{n} \cancelto{0}{p_i^2}\; z_1 \cdots z_{i-1} z_{i+1} \cdots z_n = 0\,, \end{aligned} \tag{1.494}$$

where we used momentum conservation and on-shellness. For $m = n$, we have one subsets of n elements,

$$S = A, \qquad p_S = \sum_{i=1}^{n} p_i, \qquad z_S = \prod_{i=1}^{n} z_i\,, \tag{1.495}$$

$$h_n(z,p) = \cancelto{0}{\Big(\sum_{i=1}^{n} p_i\Big)^2} \prod_{j=1}^{n} z_j\,, \tag{1.496}$$

where we used momentum conservation.

The non-vanishing polynomials $h_m(z,p)$, with $2 \leq m \leq n-2$, are homogeneous polynomials of degree m in the variables $z_1, \dots, z_n$. E.g. for $m = 2$, we have $\begin{pmatrix} n \\ 2 \end{pmatrix}$ subsets of two elements. Then

$$\begin{aligned} &S = \{i, j\}\,, \quad 1 \leq i, j \leq n\,, \quad i \neq j\,, \qquad p_S = p_i + p_j, \qquad z_S = z_i z_j\,, \\ &h_2(z,p) = \sum_{i \neq j} \,(p_i + p_j)^2\; z_i\; z_j = \sum_{ij} s_{ij} z_i z_j\,, \end{aligned} \tag{1.497}$$

The scattering equations $f_i(z,p)$ are equivalent to $h_m(z,p) = 0$, with $2 \leq m \leq n-2$.

One can rescale the polynomials as $\tilde{h}_{m-1} = \lim_{z_1 \to \infty} \frac{h_m}{z_1}$ and write $\tilde{h}_m(z,p) = 0$, with $1 \leq m \leq n{-}3$. Bezout's theorem states that the number of common solutions of a set of polynomials (is bounded, and almost always) equals the product of the degrees of the polynomials.

$$\prod_{m=1}^{n-3} \deg \tilde{h}_m = (n-3)!\;, \tag{1.498}$$

so we expect to have $(n-3)!$ inequivalent solutions of the scattering equations.

1.15.2 CHY amplitudes

Let us introduce the primed product,

$$\prod{}' \frac{1}{f_a(z,p)} = (-1)^{i+j+k}(z_i - z_j)(z_j - z_k)(z_k - z_i) \prod_{a\neq i,j,k} \frac{1}{f_a(z,p)} , \tag{1.499}$$

which is independent of the choice of i, j, k, and takes into account that only $(n-3)$ of the n scattering equations are linearly independent. Then, we introduce the PSL$(2,\mathbb{C})$ invariant measure,

$$\text{dVol}_{PSL(2,\mathbb{C})} = (-1)^{p+q+r} \frac{dz_p\; dz_q\; dz_r}{(z_p - z_q)(z_q - z_r)(z_r - z_p)} , \tag{1.500}$$

which ensures that each equivalence class of solutions is counted only once.

We introduce the short-hand $z_{ij} = z_i - z_j$. We have the overall measure,

$$\begin{aligned} d\Omega &= \frac{1}{(2\pi i)^{n-3}} \prod{}' \frac{1}{f_a(z,p)} \frac{d^n z}{d\text{Vol}_{PSL(2,\mathbb{C})}} \\ &= \frac{(-1)^{i+j+k+p+q+r}}{(2\pi i)^{n-3}}\; z_{ij}\; z_{jk}\; z_{ki}\; z_{pq}\; z_{qr}\; z_{rp} \frac{\prod\limits_{b\neq p,q,r} dz_b}{\prod\limits_{a\neq i,j,k} f_a(z,p)} . \end{aligned} \tag{1.501}$$

Under $PSL(2,\mathbb{C})$ transformations (1.480), $z^{'} = g(z)$,

$$z_i^{'} - z_j^{'} = \frac{z_i - z_j}{(cz_i + d)(cz_j + d)} \qquad \text{(see app. H.44)} , \tag{1.502}$$

$$dz_i^{'} = \frac{dz_i}{(cz_i + d)^2} , \tag{1.503}$$

so as expected the measure (1.500) is invariant under $PSL(2,\mathbb{C})$ transformations, while the measure $d^n z$ and the primed product (1.499) transform as

$$d^n z^{'} = \Big(\prod_{i=1}^{n} \frac{1}{(cz_i + d)^2}\Big)\; d^n z , \tag{1.504}$$

$$\prod{}' \frac{1}{f_a(z^{'},p)} = \Big(\prod_{i=1}^{n} \frac{1}{(cz_i + d)^2}\Big) \prod{}' \frac{1}{f_a(z,p)} . \tag{1.505}$$

Consider the $2n \times 2n$ antisymmetric matrix,

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \tag{1.506}$$

with

$$A_{ab} = \begin{cases} \dfrac{2p_a \cdot p_b}{z_a - z_b}, & a \neq b , \\ 0, & a = b . \end{cases} \qquad B_{ab} = \begin{cases} \dfrac{2\epsilon_a \cdot \epsilon_b}{z_a - z_b}, & a \neq b , \\ 0, & a = b . \end{cases}$$

$$C_{ab} = \begin{cases} \dfrac{2\epsilon_a \cdot p_b}{z_a - z_b}, & a \neq b\,, \\ -\displaystyle\sum_{j=1, j\neq a}^{n} \dfrac{2\epsilon_a \cdot p_j}{z_a - z_j} & a = b\,. \end{cases} \tag{1.507}$$

where $\epsilon_a \equiv \epsilon(p_a)$ are the polarisation vectors. The Pfaffian of Ψ vanishes (see app. H.47). However, the $(2n-2)\times(2n-2)$ matrix Ψ^{ij}_{ij} obtained by deleting rows and columns i and j has a non-vanishing Pfaffian, and we fix the polarisation factor,

$$E(p,\epsilon,z) = \frac{(-1)^{i+j}}{2^{n/2}(z_i - z_j)} \mathrm{Pf}(\Psi^{ij}_{ij})\,, \tag{1.508}$$

where the normalisation is chosen in agreement with the one of the colour matrices, $\mathrm{Tr}(T^a T^b) = \delta^{ab}$. Upon using the scattering equations (1.478), $E(p,\epsilon,z)$ does not depend on the choice of i and j.

Then, we introduce a cyclic factor,

$$C(\sigma,z) = \frac{1}{(z_{\sigma_1} - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3}) \dots (z_{\sigma_n} - z_{\sigma_1})}\,, \tag{1.509}$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of the n labels. Under a $PSL(2,\mathbb{C})$ transformation, $g(z) = z^{'}$,

$$E(p,\epsilon,z^{'}) = \prod_{i=1}^{n} (cz_i + d)^2 \; E(p,\epsilon,z)\,, \tag{1.510}$$

$$C(\sigma,z^{'}) = \prod_{i=1}^{n} (cz_i + d)^2 \; C(\sigma,z)\,. \tag{1.511}$$

Considering an integration contour ℓ that wraps around all the inequivalent zeros of the scattering equations (1.478), the colour-stripped n-gluon tree amplitude in D dimensions is given by the contour integral,

$$i\; A_n(p,\epsilon,\sigma) = i \oint_{\mathcal{C}} d\Omega\; C(\sigma,z)\; E(p,\epsilon,z)\,. \tag{1.512}$$

The n-graviton tree amplitude in D dimensions is obtained by replacing the cyclic factor with the polarisation factor,

$$i\; M_n(p,\epsilon) = i \oint_{\mathcal{C}} d\Omega\; E(p,\epsilon,z)^2\,. \tag{1.513}$$

Since the $\mathrm{Pf}(\Psi)^2 = \det(\Psi)$,

$$E(p,\epsilon,z)^2 = \frac{\det(\Psi^{ij}_{ij})}{2^n\; (z_i - z_j)^2}\,. \tag{1.514}$$

Using eqs. (1.504), (1.505), (1.510) and (1.511), we see that eqs. (1.512) and (1.513) are $PSL(2,\mathbb{C})$ invariant.

Introducing the $n \times n$ matrix Φ, with

$$\phi_{ab} = \frac{\partial f_a}{\partial z_b} = \begin{cases} \dfrac{2p_a \cdot p_b}{(z_a - z_b)^2}, & a \neq b\,, \\ -\displaystyle\sum_{j=1, j\neq a}^{n} \frac{2p_a \cdot p_j}{(z_a - z_j)^2} & a = b\,, \end{cases} \tag{1.515}$$

and the $(n-3)\times(n-3)$ matrix Φ^{ijk}_{pqr}, obtained by deleting rows i, j, k and columns p, q, r, we define

$$\det{}' \Phi = (-1)^{i+j+k+p+q+r} \frac{\left|\Phi^{ijk}_{pqr}\right|}{z_{ij}\; z_{jk}\; z_{ki}\; z_{pq}\; z_{qr}\; z_{rp}}\,. \tag{1.516}$$

Note the analogy with the $(n-3)\times(n-3)$ minor in Hodges's construction of the n-graviton MHV amplitude (1.449).

Then using the Jacobian,

$$J(z,p) = \frac{1}{\det{}'\Phi}\,, \tag{1.517}$$

we can re-write the contour integrals as sums over the inequivalent solutions $z_i^{(j)}$, with $i = 1, \ldots, n$, and $j = 1, \ldots, (n-3)!$, of the scattering equations (1.478),

$$i\; A_n(p, \epsilon, \sigma) = i \sum_{j=1}^{(n-3)!} \; J(z^{(j)}, p)\; C(\sigma, z^{(j)})\; E(p, \epsilon, z^{(j)})\,, \tag{1.518}$$

$$i\; M_n(p, \epsilon) = i \sum_{j=1}^{(n-3)!} \; J(z^{(j)}, p)\; E(p, \epsilon, z^{(j)})^2\,. \tag{1.519}$$

Under a $PSL(2, \mathbb{C})$ transformation, $g(z) = z'$,

$$J(z', p) = \Big(\prod_{i=1}^{n} \frac{1}{(cz_i + d)^4}\Big)\; J(z,p)\,, \tag{1.520}$$

thus, using eqs. (1.504), (1.505) and (1.520), each term in eqs. (1.518) and (1.519) is $PSL(2, \mathbb{C})$ invariant.

As already discussed when squaring Yang-Mills, in composing two spin-1 polarisation vectors besides considering the spin-2 polarisations, $\epsilon^{\mu\nu}_{\pm}(p) = \epsilon^{\mu}_{\pm}(p)\epsilon^{\nu}_{\pm}(p)$, we could also obtain spin-0 states. They are the dilaton and the antisymmetric tensor field (often called B field in analogy with electromagnetism),

$$\epsilon^{\mu\nu}_{dil}(p) = \frac{1}{\sqrt{2}}[\epsilon^{\mu}_{+}(p)\epsilon^{\nu}_{-}(p) + \epsilon^{\mu}_{-}(p)\epsilon^{\nu}_{+}(p)]\,, \tag{1.521}$$

$$\epsilon^{\mu\nu}_{B}(p) = \frac{1}{\sqrt{2}}[\epsilon^{\mu}_{+}(p)\epsilon^{\nu}_{-}(p) - \epsilon^{\mu}_{-}(p)\epsilon^{\nu}_{+}(p)]\,. \tag{1.522}$$

In eq. (1.519), we can have also the dilaton and the B-field as external states after replacing the cyclic factor with a polarisation factor with $\tilde{\epsilon} \neq \epsilon$,

$$i\ M_n(p, \epsilon) = i \sum_{j=1}^{(n-3)!} J(z^{(j)}, p)\ E(p, \epsilon, z^{(j)})\ E(p, \tilde{\epsilon}, z^{(j)})\,. \tag{1.523}$$

However, note that in amplitudes with only gravitons as external states, dilatons and B-fields can only appear in loops (just like supersymmetric partners do), so at tree level an n-graviton amplitude cannot tell if it comes from pure Einstein gravity or from gravity coupled to dilatons and B-fields.

The doubling procedure is reminiscent of the double copy and the CK duality we have examined in sec. 1.14.5. Further, the number $(n-3)!$ of inequivalent solutions of the scattering equations, reminds us of the number of independent colour-stripped amplitudes in an n-gluon amplitude, after using the BCJ relations in sec. 1.7.4. In fact, it is possible to show [37] that the scattering equations fulfil CK duality.

The doubling procedure works also in the reverse direction, i.e. by replacing the polarisation factor with a cyclic factor,

$$i\ A_n(p, \sigma, \tilde{\sigma}) = i \sum_{j=1}^{(n-3)!} J(z^{(j)}, p)\ C(\sigma, z^{(j)})\ C(\tilde{\sigma}, z^{(j)})\,. \tag{1.524}$$

The amplitudes $A_n(p, \sigma, \tilde{\sigma})$ are the double-ordered colour-stripped amplitudes of a bi-adjoint scalar theory with cubic vertices. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^{ab})(\partial^\mu \phi^{ab}) - \frac{\lambda}{3!} f^{a_1 a_2 a_3}\ f^{b_1 b_2 b_3} \phi^{a_1 b_1} \phi^{a_2 b_2} \phi^{a_3 b_3}\,, \tag{1.525}$$

with $a = 1, \ldots, \dim(G)$. Its amplitudes andmit a double colour decomposition in terms of $A_n(p, \sigma, \tilde{\sigma})$,

$$M(p) = \lambda^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} \sum_{\tilde{\sigma} \in S_n/\mathbb{Z}_n} \mathrm{tr}(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_n}})\ \mathrm{tr}(T^{b_{\tilde{\sigma}_1}} \cdots T^{b_{\tilde{\sigma}_n}})\ A_n(p, \sigma, \tilde{\sigma})\,. \tag{1.526}$$

1.15.3 Three- and four-gluon amplitudes

Let us consider the n-gluon scattering equation for $n = 3$. Then there is only $0! = 1$ inequivalent solution. There are only 3 variables, which can all be fixed by Möbius invariance,

$$z_1^{(1)} = 0, \qquad z_2^{(1)} = 1, \qquad z_3^{(1)} = \infty\,. \tag{1.527}$$

The minor $\left|\phi^{ijk}_{pqr}\right| = 1$ and the Jacobian (1.517) is

$$J(z, p) = (z_1 - z_2)^2\ (z_2 - z_3)^2\ (z_3 - z_1)^2\,. \tag{1.528}$$

The cyclic factor (1.509) is

$$C(\sigma, z) = \frac{1}{(z_1 - z_2)\ (z_2 - z_3)\ (z_3 - z_1)}\,. \tag{1.529}$$

The polarisation factor (1.508) is (see app. H.47)

$$E(p, \epsilon, z) = \sqrt{2}\ \frac{\epsilon_1 \cdot \epsilon_2\ \epsilon_3 \cdot p_1 + \epsilon_2 \cdot \epsilon_3\ \epsilon_1 \cdot p_2 + \epsilon_3 \cdot \epsilon_1\ \epsilon_2 \cdot p_3}{(z_1 - z_2)\ (z_2 - z_3)\ (z_3 - z_1)}\,. \tag{1.530}$$

So,

$$\begin{aligned} iA_3^{tree}(p, \epsilon, \sigma) &= iJ(z^{(1)}, p)\ C(\sigma, z^{(1)})\ E(p, \epsilon, z^{(1)}) \\ &= \sqrt{2}i\ (\epsilon_1 \cdot \epsilon_2\ \epsilon_3 \cdot p_1 + \epsilon_2 \cdot \epsilon_3\ \epsilon_1 \cdot p_2 + \epsilon_3 \cdot \epsilon_1\ \epsilon_2 \cdot p_3)\,, \end{aligned} \tag{1.531}$$

which yields the usual 3-gluon vertex.

For $n = 4$, the scattering equations are

$$\begin{aligned} \frac{s}{z_1 - z_2} + \frac{u}{z_1 - z_3} + \frac{t}{z_1 - z_4} &= 0\,, \\ \frac{s}{z_2 - z_1} + \frac{t}{z_2 - z_3} + \frac{u}{z_2 - z_4} &= 0\,, \\ \frac{u}{z_3 - z_1} + \frac{t}{z_3 - z_2} + \frac{s}{z_3 - z_4} &= 0\,, \\ \frac{t}{z_4 - z_1} + \frac{u}{z_4 - z_2} + \frac{s}{z_4 - z_3} &= 0\,, \end{aligned} \tag{1.532}$$

out of which we obtain the cross ratios,

$$\begin{aligned} \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} &= -\frac{s}{u}\,, \\ \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} &= -\frac{s}{t}\,, \\ \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} &= -\frac{u}{t}\,. \end{aligned} \tag{1.533}$$

There is one equivalent solution, which can be taken to be

$$z_1^{(1)} = -\frac{s}{t}\,, \qquad z_2^{(1)} = 0\,, \qquad z_3^{(1)} = 1\,, \qquad z_4^{(1)} = \infty\,. \tag{1.534}$$

The polynomial form of the scattering equation (1.497),

$$0 = h_2(z, p) = (z_1 z_2 + z_3 z_4)\ s + (z_2 z_3 + z_1 z_4)\ t + (z_1 z_3 + z_2 z_4)\ u\,, \tag{1.535}$$

and momentum conservation, $s + t + u = 0$, help to relate several equivalent forms of $J(z, p)$ and $E(p, \epsilon, z)$.

1.15.4 Global residue

In general, finding the solutions of the scattering equations (1.478) may be very difficult. However, there is a way of computing eqs. (1.512) and (1.513) without knowing the explicit solutions of eq. (1.478). In this section, we will follow ref. [38]. Let us re-write eq. (1.512) as

$$i\ A_n(p,\epsilon,\sigma) = i\ \frac{(-1)^{i+j+k}}{(2\pi i)^{n-3}} \oint_{\ell} \frac{d^n z}{d\text{Vol}}\ \frac{z_{ij}\ z_{jk}\ z_{ki}}{\prod\limits_{a\neq i,j,k} f_a(z,p)}\ C(\sigma,z)\ E(p,\epsilon,z)\,. \tag{1.536}$$

As we have seen, we can go from $f_a(z,p)$ to the polynomial form $h_m(z,p)$ through the Vandermonde determinant (1.490),

$$i\ A_n(p,\epsilon,\sigma) = i\ \frac{(-1)^n}{(2\pi i)^{n-3}} \oint_{\ell} \frac{d^n z}{d\text{Vol}}\ \frac{\prod\limits_{i<j}(z_i-z_j)}{\prod\limits_{m=2}^{n-2} h_m(z,p)}\ C(\sigma,z)\ E(p,\epsilon,z)\,. \tag{1.537}$$

We can use $PSL(2,\mathbb{C})$ invariance and fix three variables,

$$z_1 = 0, \qquad z_{n-1} = 1, \qquad z_n = \infty\,. \tag{1.538}$$

Then

$$d\text{Vol} = \frac{dz_1\ dz_{n-1}\ dz_n}{(z_1-z_{n-1})(z_{n-1}-z_n)(z_n-z_1)} = \frac{dz_1\ dz_{n-1}\ dz_n}{z_n^2}\,. \tag{1.539}$$

Further, we can invert z_n,

$$z_n = \frac{1}{w} \Rightarrow dz_n = -\frac{1}{w^2}\ dw = -z_n^2\ dw\,, \tag{1.540}$$

so we can write the contour integral as

$$i\ A_n(p,\epsilon,\sigma) = i\ \frac{1}{(2\pi i)^{n-3}} \oint_{\mathcal{C}} \frac{R(p,\epsilon,z,\sigma)\ dz_2\wedge\ldots\wedge dz_{n-2}}{h_2'(z,p)\cdots h_{n-2}'(z,p)}\,, \tag{1.541}$$

with

$$h_m'(z,p) = \left.\frac{dh_m(z,p)}{dz_n}\right|_{z_n=0}\,, \tag{1.542}$$

and

$$R(p,\epsilon,z,\sigma) = -\ z_n^4\ \prod_{i<j}(z_i-z_j)\ C(\sigma,z)\ E(p,\epsilon,z)\Big|_{\substack{z_1=0\\ z_{n-1}=0\\ z_n=\infty}}\,. \tag{1.543}$$

Since the polynomials $h_m(z,p)$ are linear in each variable z_a, $h_m'(z,p)$ yields the coefficient of z_n in $h_m(z,p)$. The local residue at a solution $z^{(j)}$ is defined as

$$\text{Res}(A_n)|_{z^{(j)}} = \frac{1}{(2\pi i)^{n-3}} \oint_{\Gamma_S} \frac{R(p,\epsilon,z,\sigma)\ dz_2\wedge\ldots\wedge dz_{n-2}}{h_2'(z,p)\cdots h_{n-2}'(z,p)}\,, \tag{1.544}$$

where the Γ_S is a small $(n-3)$ torus in the variables $z_2, \ldots, z_{n-2}$ around $z^{(j)}$, with orientation,

$$\mathrm{d}\,\arg(h_2) \wedge \mathrm{d}\,\arg(h_3) \wedge \ldots \wedge \mathrm{d}\,\arg(h_{n-2}) \geq 0\,. \tag{1.545}$$

Then, we can define the global residue as

$$\mathrm{GRes}(A_n) = \sum_{j=1}^{(n-3)!} \mathrm{Res}(A_n)|_{z^{(j)}}\,. \tag{1.546}$$

The ratio $\dfrac{R(p,\epsilon,z,\sigma)}{h_2'(z,p)\cdots h_{n-2}'(z,p)}$ is defined in a finite-dimensional vector space. Let us suppose that a basis in this vector space is $\{e_i\}$, and let us consider two polynomials P_1 and P_2 in the vector space.

In algebraic geometry, the global residue of the product of two polynomials defines a symmetric inner product $\mathrm{GRes}(P_1 \cdot P_2) = \langle P_1, P_2\rangle$. Since the product is non-degenerate, there must be a dual basis Δ_i, such that $\langle e_i, \Delta_j\rangle = \delta_{ij}$. In order to compute the global residue of one polynomial P, one can decompose the polynomial on the $\{e_i\}$ basis, $P = \sum_i a_i e_i$, and decompose the identity on the dual basis, $\mathbb{1} = \sum_i b_i\, \Delta_i$. Then

$$\mathrm{GRes}(P) = \mathrm{GRes}(P \cdot \mathbb{1}) = \sum_i a_i\, b_i\,. \tag{1.547}$$

This allows one to compute the global residue of a polynomial without knowing the solutions to the scattering equations. Now, $R(p,\epsilon,z,\sigma)$ is a rational function, but it can be put in polynomial form.

The procedure sketched above was introduced in ref. [39], and is reviewed in ref. [38].

1.16 Amplitudes for all masses and spins

In this section, we want to consider general properties of amplitudes with particles of arbitrary mass and spin. In doing that, we will realise that the little group is even more important than what we could surmise from its scaling.

As we said in sec. 1.2.3, since

$$\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} 0 & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\lambda}^{\dot{a}}(p)\lambda^a(p) \\ \lambda_a(p)\tilde{\lambda}_{\dot{a}}(p) & 0 \end{pmatrix} \tag{1.548}$$

the little group $U(1)$ transformation,

$$\lambda_a \to e^{i\phi/2}\lambda^a \qquad \tilde{\lambda}_{\dot{a}} \to e^{-i\phi/2}\tilde{\lambda}_{\dot{a}}\,. \tag{1.549}$$

leaves $\not{p}$ invariant. For complex momenta, $\lambda_a \to t\lambda_a$ with t a complex number, and we can take the little group as $GL(1)$.

Taking as a reference momentum,

$$k^\mu = (E, 0, 0, E) = (2E, 0; 0, 0)\,, \tag{1.550}$$

where the right-most expression is in light-cone coordinates, eq. (1.65) implies that

$$\lambda_a(k) = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \tilde{\lambda}_{\dot{a}}(k) = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}, \tag{1.551}$$

$$k_{a\dot{a}} = 2E\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \tag{1.552}$$

then rotations about the z axis are

$$\Lambda^b{}_a = \begin{pmatrix}e^{i\phi/2} & 0\\0 & e^{-i\phi/2}\end{pmatrix}, \qquad \tilde{\Lambda}^{\dot{b}}{}_{\dot{a}} = \begin{pmatrix}e^{-i\phi/2} & 0\\0 & e^{i\phi/2}\end{pmatrix}, \tag{1.553}$$

and we get eq. (1.549).

Note that using Euler angles θ, ϕ

$$\begin{aligned}
p^\mu &= (E, E\sin(\theta)\cos(\phi), E\sin(\theta)\sin(\phi), E\cos(\theta))\,,\\
p^+ &= p^0 + p^z = E(1+\cos(\theta)) = 2E\cos^2(\theta/2)\,,\\
p_\perp &= p^x + ip^y = E\sin(\theta)(\cos(\phi) + i\sin(\phi)) = 2E\sin(\theta/2)\cos(\theta/2)e^{i\phi}\,,
\end{aligned} \tag{1.554}$$

so

$$\frac{p_\perp}{\sqrt{p^+}} = \sqrt{2E}\sin(\theta/2)e^{i\phi}\,, \tag{1.555}$$

the spinors (1.65) and (1.71) can be taken as

$$\xi_+(p) = \lambda_a(p) = \frac{1}{\sqrt{p^+}}\begin{pmatrix}p^+\\p_\perp\end{pmatrix} = \sqrt{2E}\begin{pmatrix}c\\s\end{pmatrix}, \tag{1.556}$$

$$\xi_+^\dagger(p) = \tilde{\lambda}_{\dot{a}}(p) = \frac{1}{\sqrt{p^+}}\begin{pmatrix}p^+\\p_\perp^*\end{pmatrix} = \sqrt{2E}\begin{pmatrix}c\\s^*\end{pmatrix}, \tag{1.557}$$

$$\xi_-(p) = \tilde{\lambda}^{\dot{a}}(p) = \frac{1}{\sqrt{p^+}}\begin{pmatrix}-p_\perp^*\\p_+\end{pmatrix} = \sqrt{2E}\begin{pmatrix}-s^*\\c\end{pmatrix}, \tag{1.558}$$

$$\xi_-^\dagger(p) = \lambda^a(p) = \frac{1}{\sqrt{p^+}}\begin{pmatrix}-p_\perp\\p_+\end{pmatrix} = \sqrt{2E}\begin{pmatrix}-s\\c\end{pmatrix}. \tag{1.559}$$

with $c = \cos(\theta/2)$, $s = \sin(\theta/2)e^{i\phi}$. As we saw in sec. 1.2.3,

$$p_{a\dot{a}} = \lambda_a(p)\tilde{\lambda}_{\dot{a}}(p) = 2E\begin{pmatrix}c^2 & cs^*\\cs & ss^*\end{pmatrix}, \tag{1.560}$$

has rank 1, since $\det(\lambda_a(p)\tilde{\lambda}_{\dot{a}}(p)) = 0$.

In sec. 1.2.3, we also saw that for massive particles, $p_{a\dot{a}}$ has rank 2, and we can write it as the sum of two rank 1 matrices,

$$p_{a\dot{a}} = \lambda_a{}^1(p)\tilde{\lambda}_{\dot{a}1}(p) + \lambda_a{}^2(p)\tilde{\lambda}_{\dot{a}2}(p)\,, \tag{1.561}$$

where $I = 1, 2$ in $\lambda_a{}^I(p)$ labels the spin 1/2 representation of the little group $SU(2)$. However, the $\lambda_a{}^I(p)$ are not uniquely associated to a given momentum, since we may perform the $SU(2)$ transformation,

$$\lambda_a{}^I \to W^I{}_J\lambda_a{}^J \qquad \tilde{\lambda}_{\dot{a}}{}^I \to (W^{-1})^I{}_J\tilde{\lambda}_{\dot{a}}{}^J\,. \tag{1.562}$$

For massive particles of spin $S > 1/2$, we may label states of spin S as symmetric tensors of rank $2S$ (see app. H.40 and ref. [33]). Then amplitudes for spin-S massive particles are Lorentz-invariant functions of symmetric rank-$2S$ tensors.

We raise and lower indices with the $SU(2)$ antisymmetric tensor (1.73), with $\epsilon^{ab}\epsilon_{bc} = \delta^a{}_c$. Now,

$$p_{a\dot{a}} = \lambda_a{}^I(p)\tilde{\lambda}_{\dot{a}I}(p) = \epsilon_{IJ}\lambda_a{}^I(p)\tilde{\lambda}_{\dot{a}}{}^J(p)\,, \tag{1.563}$$

and since $\epsilon^{\alpha\beta}A_{\alpha\gamma}A_{\beta\delta} = \det(A)\epsilon_{\gamma\delta}$, we can formally relate $\tilde{\lambda}^{\dot{a}I}(p)$ to $\lambda_a{}^I$,

$$\begin{aligned} p_{a\dot{a}}\tilde{\lambda}^{\dot{a}I} &= \epsilon_{KJ}\lambda_a{}^K\tilde{\lambda}_{\dot{a}}{}^J\tilde{\lambda}^{\dot{a}I} = \epsilon_{KJ}\lambda_a{}^K\tilde{\lambda}^J_{\dot{a}}\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{b}}{}^I = \epsilon_{KJ}\lambda_a{}^K\epsilon^{JI}\det(\tilde{\lambda}) = \det(\tilde{\lambda})\cdot\lambda_a{}^I\,, \\ p_{a\dot{a}}\tilde{\lambda}^{aI} &= \epsilon_{KJ}\lambda_a{}^K\tilde{\lambda}_{\dot{a}}{}^J\lambda^{aI} = \epsilon_{KJ}\lambda_a{}^K\tilde{\lambda}^J_{\dot{a}}\epsilon^{ab}\lambda_b{}^I = \epsilon_{KJ}\tilde{\lambda}_{\dot{a}}{}^J\det(\lambda)\epsilon^{KI} = -\det(\lambda)\cdot\tilde{\lambda}_{\dot{a}}{}^I\,. \end{aligned} \tag{1.564}$$

Further,

$$\begin{aligned} p_{a\dot{a}}p^{a\dot{a}} &= \epsilon_{KJ}\lambda_a{}^K\tilde{\lambda}_{\dot{a}}{}^J\lambda^{aI}\tilde{\lambda}^{\dot{a}L}\epsilon_{IL} \\ &= \epsilon_{KJ}\lambda_a{}^K\tilde{\lambda}_{\dot{a}}{}^J\epsilon^{ab}\lambda_b{}^I\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{b}}{}^L\epsilon_{IL}\,, \\ &= \epsilon_{KJ}\det(\lambda)\epsilon^{KI}\det(\tilde{\lambda})\epsilon^{JL}\epsilon_{IL} \\ &= \det(\lambda)\det(\tilde{\lambda})\delta_K{}^L\delta^K{}_L \\ &= 2\det(\lambda)\det(\tilde{\lambda})\,. \end{aligned} \tag{1.565}$$

As a reference momentum, we take $k^\mu = (m, 0, 0, 0)$, then we can perform a boost to the momentum,

$$p^\mu = (E, |\bar{p}|\sin(\theta)\cos(\phi), |\bar{p}|\sin(\theta)\sin(\phi), |\bar{p}|\cos(\theta))\,, \tag{1.566}$$

with $p^2 = E^2 - |\bar{p}|^2 = m^2$ (see app. H.40 and ref. [33]).

We can expand ${\lambda_a}^I$ in a basis of two-dimensional spinors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ in the little-group space,

$$\begin{aligned}{\lambda_a}^I(p) &= \begin{pmatrix}\sqrt{E+p}\,c & -\sqrt{E-p}\,s^* \\ \sqrt{E+p}\,s & \sqrt{E-p}\,c\end{pmatrix} \\ &= \lambda_a(p)\otimes\begin{pmatrix}1\\0\end{pmatrix} + \eta_a(p)\otimes\begin{pmatrix}0\\1\end{pmatrix}\,, \end{aligned} \tag{1.567}$$

with

$$\lambda_a(p) = \sqrt{E+p}\begin{pmatrix}c\\s\end{pmatrix}\,, \quad \eta_a(p) = \sqrt{E-p}\begin{pmatrix}-s^*\\c\end{pmatrix}\,. \tag{1.568}$$

Likewise,

$$\begin{aligned}{\tilde\lambda_{\dot a}}{}^I(p) &= \begin{pmatrix}\sqrt{E-p}\,s & \sqrt{E+p}\,c \\ -\sqrt{E-p}\,c & \sqrt{E+p}\,s^*\end{pmatrix} \\ &= \tilde\lambda_{\dot a}(p)\otimes\begin{pmatrix}0\\1\end{pmatrix} - \tilde\eta_{\dot a}(p)\otimes\begin{pmatrix}1\\0\end{pmatrix}\,, \end{aligned} \tag{1.569}$$

with

$$\tilde\lambda_{\dot a}(p) = \sqrt{E+p}\begin{pmatrix}c\\s^*\end{pmatrix}\,, \qquad \tilde\eta_{\dot a}(p) = \sqrt{E-p}\begin{pmatrix}-s\\c\end{pmatrix}\,. \tag{1.570}$$

Lowering the little-group index,

$$\begin{aligned}\tilde\lambda_{\dot aI}(p) &= \epsilon_{IJ}{\tilde\lambda_{\dot a}}{}^J(p) \\ &= \tilde\lambda_{\dot a}(p)\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \tilde\eta_{\dot a}(p)\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\ &= \tilde\lambda_{\dot a}(p)\otimes\begin{pmatrix}1\\0\end{pmatrix} + \tilde\eta_{\dot a}(p)\otimes\begin{pmatrix}0\\1\end{pmatrix}\,, \end{aligned} \tag{1.571}$$

so that using eqs. (1.567) and (1.571), we obtain a representation of eq. (1.563),

$$\begin{aligned}p_{a\dot a} &= {\lambda_a}^I(p)\tilde\lambda_{\dot aI}(p) \\ &= \lambda_a(p)\tilde\lambda_{\dot a}(p) + \eta_a(p)\tilde\eta_{\dot a}(p)\,, \end{aligned} \tag{1.572}$$

as the explicit sum of two rank 1 matrices.

Note that

$$\det(\lambda) = \det(\tilde\lambda) = \sqrt{E^2-p^2}\;(c^2+ss^*) = m\,, \tag{1.573}$$

so that eqs. (1.564) and (1.565) imply that $p_{a\dot a}\tilde\lambda^{\dot aI} = m{\lambda_a}^I$, $p_{a\dot a}\lambda^{aI} = -m{\tilde\lambda_{\dot a}}{}^I$ and $p_{a\dot a}p^{a\dot a} = 2m^2$.

Furthermore,

$$\langle\lambda(p)\eta(p)\rangle = -\lambda_a\epsilon^{ab}\eta_b = -\sqrt{E+p}\sqrt{E-p}\;(c \quad s)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-s^*\\ c\end{pmatrix} = m\,, \tag{1.574}$$

$$[\tilde\lambda(p)\tilde\eta(p)] = \tilde\lambda_{\dot a}\epsilon^{\dot a\dot b}\tilde\eta_{\dot b} = \sqrt{E+p}\sqrt{E-p}\;(c \quad s^*)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-s\\ c\end{pmatrix} = -m\,, \tag{1.575}$$

i.e. $\langle\lambda\eta\rangle = [\tilde\eta\tilde\lambda] = m$, which vanishes in the massless limit, as expected.

Note that in the high-energy limit,

$$E+p \to 2E, \qquad \sqrt{E-p} = \frac{m}{\sqrt{E+p}} \to \frac{m}{\sqrt{2E}}\,, \tag{1.576}$$

$$\lambda_a{}^I(p) \to \lambda_a(p) + \mathcal{O}\Big(m/\sqrt{E}\Big)\,, \tag{1.577}$$

$$\tilde\lambda_{\dot a I}(p) \to \tilde\lambda_{\dot a}(p) + \mathcal{O}\Big(m/\sqrt{E}\Big)\,, \tag{1.578}$$

i.e. they are reduced to the usual λ_a and $\tilde\lambda_{\dot a}$ spinors (1.556) and (1.557) of the massless case.

Chapter 2

Loop Amplitudes

2.1 Unitarity

2.1.1 The optical theorem

The S matrix is defined as

$$S = \mathbb{1} + i\,T\,, \tag{2.1}$$

where T represents its non-trivial part,

$$\langle f|\,T\,|i\rangle = (2\pi)^4\ \delta^4(p_f - p_i)\ M(i \to f)\,, \tag{2.2}$$

where $p_i(p_f)$ are the momenta of the initial (final) states.

Unitarity implies that

$$\mathbb{1} = S^\dagger\ S = (\mathbb{1} - i\,T^\dagger)(\mathbb{1} + i\,T) = \mathbb{1} + i\ (T - T^\dagger) + T^\dagger\ T\,, \tag{2.3}$$

thus,

$$-i\ (T - T^\dagger) = T^\dagger\ T\,. \tag{2.4}$$

Using the completeness of the Hilbert space over one-particle and multi-particle states,

$$\sum_n \int d\Pi_n\ \langle X_n|X_n\rangle = \mathbb{1}\,, \tag{2.5}$$

where $d\Pi_n$ is a short-hand for

$$d\Pi_n \equiv \prod_{j=1}^{n} \frac{dp_j^3}{(2\pi)^3\ (2E_j)}\,, \tag{2.6}$$

in the unitarity relation (2.4), we obtain the optical theorem,

$$M(i \to f) - M^*(f \to i) = i\ \sum_n \int dPS_n\ M(i \to X_n) M^*(f \to X_n)\,, \tag{2.7}$$

where the n-body phase space is

$$\mathrm{dPS}_n = d\Pi_n\ (2\pi)^4\ \delta^4(p_i - p_{x_n})\,, \qquad \text{with } p_{x_n} = \sum_{j=1}^{n}\ p_j\,. \tag{2.8}$$

In particular, if $|i\rangle = |f\rangle = |A\rangle$ from the unitarity relation (2.4), written as

$$2\ \mathrm{Im}(T) = T^\dagger\ T\,, \tag{2.9}$$

we obtain

$$2\ \mathrm{Im}\big(M(A \to A)\big) = \sum_n \int \mathrm{dPS}_n\ |M(A \to X_n)|^2\,. \tag{2.10}$$

The optical theorem relates amplitudes on the left-hand side to squares of amplitudes on the right-hand side. In its most general (2.7) or specific $|i\rangle = |f\rangle$ (2.10) sense, it is a non-perturbative statement. As such, it must hold order by order in perturbation theory. When used in a perturbative expansion of the amplitude in the coupling, it relates higher-order terms on the left-hand side to lower-order terms on the right-hand side.

Let us take e.g. $4-, 5-$ and $6-$ gluon amplitudes as expansions in the strong coupling constant g,

$$M_4 = g^2 M_4^{(0)} + g^4 M_4^{(1)} + g^6 M_4^{(2)} + \ldots\,, \tag{2.11}$$

$$M_5 = g^3 M_5^{(0)} + g^5 M_5^{(1)} + g^7 M_5^{(2)} + \ldots\,, \tag{2.12}$$

$$M_5 = g^4 M_6^{(0)} + g^6 M_6^{(1)} + g^8 M_6^{(2)} + \ldots\,. \tag{2.13}$$

We expand out the amplitudes in the optical theorem (2.10), and we write down the coefficients at order g^2, g^4, g^6 and g^8,

$$2\ \mathrm{Im}\left(M_4^{(0)}\right) = 0\,, \tag{2.14}$$

$$2\ \mathrm{Im}\big(M_4^{(1)}\big) = \int \mathrm{dPS}_2\ M_4^{(0)\dagger} M_4^{(0)}\,, \tag{2.15}$$

$$2\ \mathrm{Im}\big(M_4^{(2)}\big) = \int \mathrm{dPS}_2\ \big(M_4^{(1)\dagger} M_4^{(0)} + M_4^{(0)\dagger} M_4^{(1)}\big) + \int \mathrm{dPS}_3\ M_5^{(0)\dagger} M_5^{(0)}\,, \tag{2.16}$$

$$\begin{aligned} 2\ \mathrm{Im}\big(M_4^{(3)}\big) &= \int \mathrm{dPS}_2\ \big(M_4^{(2)\dagger} M_4^{(0)} + M_4^{(0)\dagger} M_4^{(2)} + M_4^{(1)\dagger} M_4^{(1)}\big) \\ &+ \int \mathrm{dPS}_3\ \big(M_5^{(1)\dagger} M_5^{(0)} + M_5^{(0)\dagger} M_5^{(1)}\big) + \int \mathrm{dPS}_4\ M_6^{(0)\dagger} M_6^{(0)}\,. \end{aligned} \tag{2.17}$$

Eq. (2.14) states that tree amplitudes are real (taken as functions of real momenta). Eq. (2.15) states that the imaginary part of the one-loop amplitude is related to the product of tree amplitudes, whose intermediate state X_2 is a two-particle cut. Diagramatically,

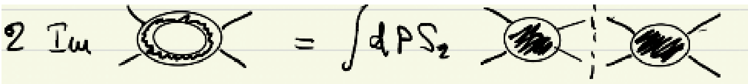


Figure 2.1: Relation of the imaginary part of the one-loop amplitude to the product of tree amplitudes.

In eq. (2.16), also the three-particle cut appears. In eq. (2.17), the four-particle cut appears and so on.

Considering now Feynman diagrams, the imaginary part of a Feynman propagator can be written as

$$\mathrm{Im}\Big(\frac{1}{p^2 - m^2 + i\epsilon}\Big) = \frac{1}{2i}\Big(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon}\Big)\,. \tag{2.18}$$

If we write the propagator through the principal value P,

$$\frac{1}{p^2 - m^2 \pm i\epsilon} = P\Big(\frac{1}{p^2 - m^2}\Big) \mp i\pi\delta(p^2 - m^2)\,, \tag{2.19}$$

then we get

$$\mathrm{Im}\Big(\frac{1}{p^2 - m^2 + i\epsilon}\Big) = -\pi\delta(p^2 - m^2)\,, \tag{2.20}$$

i.e. the propagator is real, except where it vanishes, that is when a particle goes on-shell. Thus, an amplitude is real, unless some propagators vanish. So, the imaginary part of loop amplitudes must come from intermediate states going on-shell. This is in agreement with what we found from the optical theorem.

In sec. 1.12.4, we have mentioned that in a unitary theory, poles of Green's functions, and so of amplitudes, correspond to the exchange of on-shell intermediate states. This means that single-particle states and bound states (like e.g. positronium in the amplitude of $e^+ e^-$ scattering) appear as isolated poles.

Let us consider now the amplitude as a complex function of its momenta, in particular let us examine its behaviour as a function of the Mandelstam invariant $s_{12} = (p_1 + p_2)^2$ for two particles of masses m_1 and m_2.

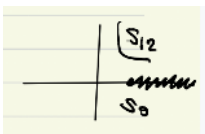


Figure 2.2: s_{12} complex plane, with branch point at s_0.

$s_0 = (m_1 + m_2)^2$ is the threshold for the creation of the two particles. For $s_{12} < s_0$, intermediate states cannot go on-shell, so the amplitude is real, and we can write

$$M(s_{12}) = (M(s_{12}^*))^*\,. \tag{2.21}$$

Then we can continue M to the whole complex s_{12} plane. For $s_{12} > s_0$, eq. (2.21) implies that

$$\begin{aligned} \text{Re}\,\big(M(s_{12}+i\epsilon)\big) &= \text{Re}\,\big(M(s_{12}-i\epsilon)\big)\,, \\ \text{Im}\,\big(M(s_{12}+i\epsilon)\big) &= -\text{Im}\,\big(M(s_{12}-i\epsilon)\big)\,, \end{aligned} \tag{2.22}$$

So there is a branch cut starting at s_{12} and we can define the discontinuity across the cut as

$$\text{Disc}\,\big(M(s_{12})\big) = 2i\ \text{Im}\,\big(M(s_{12}+i\epsilon)\big)\,. \tag{2.23}$$

A consequence for massless particles is that branch points are located at vanishing values of the Mandelstam invariants. Eqs. (2.14)-(2.17) can be re-read as

$$\text{Disc}\,\big(M_4^{(0)}\big) = 0\,, \tag{2.24}$$

$$\text{Disc}\,\big(M_4^{(1)}\big) = i\int \text{dPS}_2\ M_4^{(0)\dagger} M_4^{(0)} \qquad \text{and so on}\,. \tag{2.25}$$

Eq. (2.24) states that tree amplitudes have no branch cuts.

2.1.2 Feynman tree theorem

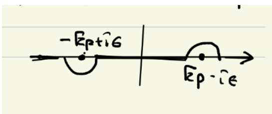
A final comment about propagators: a particle goes on-shell when $p^2 = (p_0)^2 - |\vec{p}|^2 = m^2$, i.e. when $p^0 = \pm E_p$, with $E_p = \sqrt{|p|^2 + m^2}$. Writing,

$$p^2 - m^2 + i\epsilon = (p^0)^2 - E_p^2 + i\epsilon = (p^0 - E_p + i\epsilon)\ (p^0 + E_p - i\epsilon)\,, \tag{2.26}$$

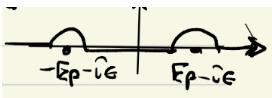
we can write Feynman's propagator as

$$D_F = \frac{i}{p^2 - m^2 + i\epsilon} = \frac{i}{2E_p}\Big(\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 + E_p - i\epsilon}\Big)\,, \tag{2.27}$$

corresponding to the poles

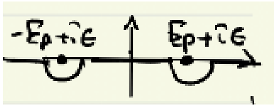


From QFT1, we know that a retarded propagator has poles below the real axis at $p^0 = \pm E_p - i\epsilon$,



$$D_R = \frac{i}{2E_p}\Big(\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 + E_p + i\epsilon}\Big)\,. \tag{2.28}$$

Conversely, an advanced propagator has poles above the real axis at $p^0 = \pm E_p + i\epsilon$,



$$D_A = \frac{i}{2E_p}\Big(\frac{1}{p^0 - E_p - i\epsilon} - \frac{1}{p^0 + E_p - i\epsilon}\Big)\,, \tag{2.29}$$

So we can write Feynman's propagator in terms of either an advanced or a retarded propagator, e.g.

$$D_F = D_A + \frac{i}{2E_p}\Big(\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 - E_p - i\epsilon}\Big)\,. \tag{2.30}$$

Using

$$\frac{1}{p^0 - E_p \pm i\epsilon} = P\Big(\frac{1}{p^0 - E_p}\Big) \mp i\pi\delta(p^0 - E_p)\,, \tag{2.31}$$

we can write Feynman's propagator as

$$D_F = D_A + \frac{\pi}{E_p}\,\delta(p^0 - E_p)\,. \tag{2.32}$$

In a loop amplitude, one can replace Feynman's propagators with eq. (2.32), and then realise that the term proportional to D_A's only drops out of the integral (e.g. see the one-loop two-point function in sec. 24.1 of ref. [5]). One can then replace back $D_A = D_F - \delta$, such that all the terms of the integrand are products of Feynman's propagator and at least one δ function. That is, one has decomposed the loop amplitude into tree amplitudes. Feynman tree theorem states that this procedure can always be implemented.

2.2 One-loop amplitudes

One-loop n-point amplitudes admit trace based and multiperipheral colour decompositions that we review shortly in the Appendix D. Using them, we limit ourselves to discuss colour-stripped amplitudes $A^{[1]}_{n;1}(1,\ldots,n)$.

In $D = 4 - 2\epsilon$ dimensions, a one-loop n-point amplitude can be reduced to scalar integrals with four, three and two internal propagators (and also evantually five, for the 2ϵ dimensions),

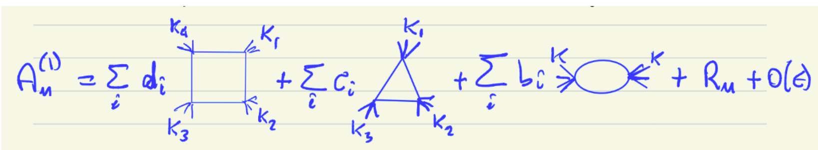


Figure 2.3: Decomposition of a one-loop amplitude into scalar integrals.

$$I_4(K_1, K_2, K_3, K_4) = \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \, \frac{1}{\ell^2(\ell - K_1)^2(\ell - K_1 - K_2)^2(\ell + K_4)^2} \, , \tag{2.33}$$

$$I_3(K_1, K_2, K_3) = \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \, \frac{1}{\ell^2(\ell - K_1)^2(\ell + K_3)^2} \, , \tag{2.34}$$

$$I_2(K) = \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \, \frac{1}{\ell^2(\ell - K)^2} \, . \tag{2.35}$$

where the K_i represent the sums over the partitions of the external momenta into four, three, or two sets, one per vertex. The coefficients b_i, c_i, d_i do not depend on ϵ. R_n is a rational part, that cannot be obtained from cuts in four dimensions, as we will discuss later. In the case of massless propagators, *tadpole* integrals, i.e. with one propagator, vanish in dimensional regularisation, but they occur if the propagator is massive.

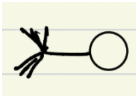


Figure 2.4: Tadpole integral.

Because of the 2ϵ dimensions, also *pentagon* integrals, i.e. with five propagators, may appear.

The *boxes*, i.e. the scalar integrals with four internal propagators, can be further decomposed as

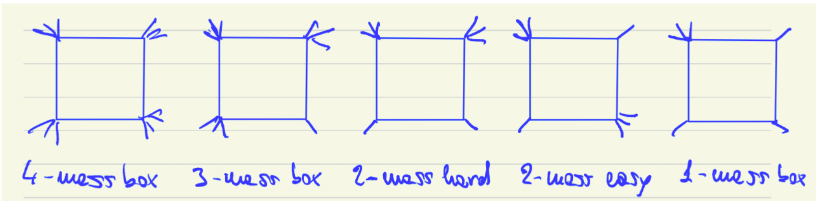


Figure 2.5: Box scalar integrals which may contribute to the one-loop amplitude.

If each K_i contains at least two particles, such that $K_i^2 > 0$, the box is termed a four-mass box, since each vertex is characterised by a time-like K_i^2, as for a massive particle. If one K contains only one particle, such that $K^2 = 0$, the box is termed a three-mass box. If there are two K's with only one particle, we have two-mass boxes (*easy* or *hard* according to the locations of the one-particle K's). If only one K contains at least two particles, we have a one-mass box.

So the one-loop n-point amplitude decomposition is

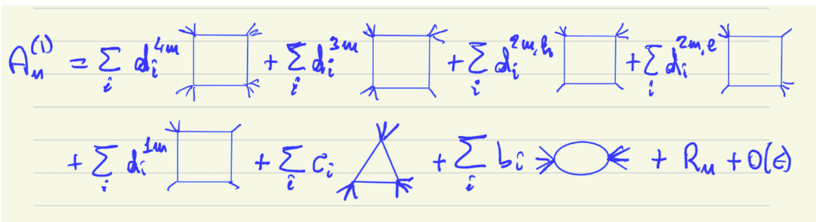
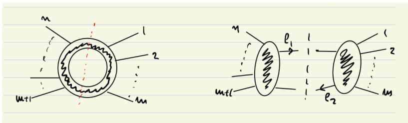


Figure 2.6: Decomposition of a one-loop amplitude into scalar integrals, including the explicit decomposition of the box integrals.

The computation of the one-loop n-point amplitudes is then reduced to the problem of the computation of the coefficients b_i, c_i, d_i and the rational part of R_n.

2.3 The unitarity method

In deriving the optical theorem from unitarity as described above, we assumed that the intermediate states are on shell, with real momenta and positive energies. As we saw, this implies that the imaginary part, or the discontinuity, of an amplitude can be evaluated through two-particle cuts.



Let us consider the discontinuity in the channel $s_{1\ldots m} = P_{1,m}^2$, with $P_{i,j} = p_i + \ldots + p_j$. Through the two-particle cut, we can write

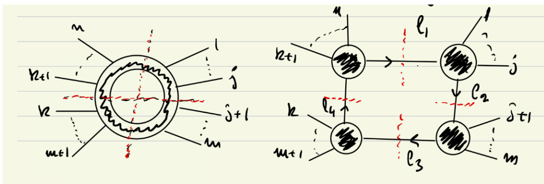
$$\begin{aligned}\mathrm{Disc}|_{s_{1\ldots m}}(A_{n;1}^{[1]}) \;&=\; i\,(2\pi)^2 \int \frac{\mathrm{d}^D\ell_1}{(2\pi)^D}\,\delta^+(\ell_1)\;A_{m+2}^{(0)}(-\ell_1^{h_1}, p_1, \ldots, p_m, \ell_2^{h_2}) \\ &\quad\cdot\,\delta^+(-\ell_2)\;A_{n-m+2}^{(0)}(-\ell_2^{-h_2}, p_{m+1}, \ldots, p_n, \ell_1^{h_1})\,, \qquad (2.36)\end{aligned}$$

where $\ell_2 = \ell_1 - P_{1,m}$, and $\delta^+(k) = \delta(k^2)\;\theta(k^0)$, which enforces that the intermediate states are on-shell, with real momenta and positive energies. The loop momenta with the two δ-function, which fix $\ell_1^2 = \ell_2^2 = 0$, yields the two-body phase space typical of two-particle production.

The idea of the unitary method is that the information from the unitarity cuts can be compared with the cuts of the one-loop decomposition, in order to determine the coefficients b_i, c_i, d_i. Generalised unitarity consists in relaxing the conditions on the intermediate states, which need not have real momenta or positive energies. Then, up to four cuts and so four intermediate states are possible. In four dimensions, no more than four cuts are possible because each cut entails a condition of the form $(\ell - K_i)^2 = 0$. So four cuts suffice to determine the four components of the loop momentum ℓ^μ.

2.3.1 The quadruple cut

Let us consider the quadruple cut. Triangles and bubbles do not contribute to it, since they do not have four propagators to cut. Thus, quadruple cuts are useful to determine the coefficients d_i. Further, each box is in a one-to-one correspondence with a specific quadruple cut, because they are characterised by the same partition of the momenta into four sets K_1, K_2, K_3, K_4, with $K_1 + K_2 + K_3 + K_4 = 0$. Let us partition the momenta as in the figure below,



The discontinuity through the quadruple cut is

$$\mathrm{Disc}_{LS}(A^{[1]}_{n;1}) = i\,(2\pi)^4 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\, \delta^+(\ell_1^2)\, \delta^+(\ell_2^2)\, \delta^+(\ell_3^2)\, \delta^+(\ell_4^2)\, A_1^{(0)}\, A_2^{(0)}\, A_3^{(0)}\, A_4^{(0)}\,, \tag{2.37}$$

where the four cut loop momenta are

$$\ell_1\;; \qquad \ell_2 = \ell_1 - P_{1,j}\;; \qquad \ell_3 = \ell_2 - P_{j+1,m}\;; \qquad \ell_4 = \ell_3 - P_{m+1,k} = \ell_1 + P_{k+1,n}\,, \tag{2.38}$$

with $K_1 = P_{1,j}$, $K_2 = P_{j+1,m}$, $K_3 = P_{m+1,k}$, $K_4 = P_{k+1,n}$, each cut imposing a constraint,

$$\ell_1^2 = \ell_2^2 = \ell_3^2 = \ell_4^2 = 0\,, \tag{2.39}$$

with

$$\begin{aligned} A_1^{(0)} &= (-\ell_1, P_{1,j}, \ell_2) \qquad A_2^{(0)} = (-\ell_2, P_{j+1,m}, \ell_3)\,,\\ A_3^{(0)} &= (-\ell_3, P_{m+1,k}, \ell_4) \qquad A_4^{(0)} = (-\ell_4, P_{k+1,n}, \ell_1)\,. \end{aligned} \tag{2.40}$$

If we take the differences,

$$\ell_1^2 - \ell_2^2 = \ell_2^2 - \ell_3^2 = \ell_3^2 - \ell_4^2 = 0\,, \tag{2.41}$$

three of the constraints become linear,

$$2\,\ell_1\cdot P_{1,j} = P_{1,j}^2\;; \qquad 2\,\ell_2\cdot P_{j+1,m} = P_{j+1,m}^2\;; \qquad 2\,\ell_3\cdot P_{m+1,k} = P_{m+1,k}^2\,. \tag{2.42}$$

One can solve the four equations for the four components of ℓ_1^μ. The three linear equations have a unique solution, $\ell_1^2 = 0$ provides at most two solutions. So one can expect at most two discrete solutions, $d_\pm^{4m}$, for ℓ_1^μ. Then ℓ_2^μ, ℓ_3^μ, ℓ_4^μ are determined by eqs. (2.40). So the two solutions are given by

$$d_\pm^{4m} = A_1^{(0)}(\ell^\pm)\; A_2^{(0)}(\ell^\pm)\; A_3^{(0)}(\ell^\pm)\; A_4^{(0)}(\ell^\pm)\,. \tag{2.43}$$

The discontinuity through the quadruple cut is then given by

$$\mathrm{Disc}_\pm\big(A^{[1]}_{n:1}\big) = d_\pm^{4m}\; I^{4m}\,. \tag{2.44}$$

Up to a Jacobian, which comes from converting the loop integral $\mathrm{d}^4\ell$ over the real part of $\mathbb{C}^4$ to an integral over the four contours which encircle the propagator poles in the quadruple cut, but which is immaterial because it cancels out on the two sides of the equation for $\mathrm{Disc}_\pm\big(A^{[1]}_{n:1}\big)$, the solutions $d_\pm^{4m}$ define the leading singularities of $A^{[1]}_{n:1}$ (although, in spite of the name, the $d_\pm^{4m}$ are not singular at all). The solution is then taken to be

$$d^{4m} = \frac{d_+^{4m} + d_-^{4m}}{2}\,, \tag{2.45}$$

such that

$$\mathrm{Disc}_{LS}\big(A_{n;1}^{[1]}\big) = d^{4m}\ I^{4m}\,. \tag{2.46}$$

The solution for generic values of the P's, and thus for a generic four-mass box, is provided in ref. [40].

It is convenient to consider specific helicity configurations, and use the counting of (negative) helicities as a selection rule on the mass boxes which may contribute. For example, let us consider the quadruple cut of the four-mass box of a NMHV amplitude in fig. 2.7,

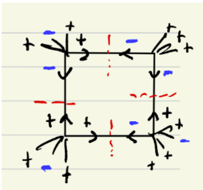


Figure 2.7: Quadruple cut of a four-mass box of a NMHV amplitude, where we have labelled in blue the negative helicities.

We see that the tree amplitude in the lower left vertex has only one negative helicity, and vanishes. The reason is that there are four tree amplitudes, all with more than three legs, so there must be eight negative helicities, while the four-cut NMHV amplitude only has seven, four from the cuts and three from the external legs. We can repeat the counting for a MHV amplitude with the same conclusion.

Thus, for MHV and NMHV amplitudes, the four-mass box coefficients d^{4m} vanish. They contribute to amplitudes which have at least four negative helicities, thus at NNMHV level and beyond.

Likewise, for MHV amplitudes three-mass box coefficients d^{3m} vanish, because there must be at least seven negative helicities (the tree three-point amplitude only needs one), but there can be only six, four from the cuts and two from the external states, as in fig. 2.8.

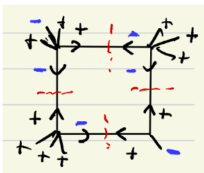


Figure 2.8: Quadruple cut of a three-mass box of a MHV amplitude.

Two-mass boxes contribute to MHV amplitudes, however the coefficient $d^{2m,h}$ of the *hard* one vanishes, for which the massive K are adjacent. We can see it through a triple cut which puts the massless vertices into one tree amplitude, as in fig. 2.9. There are three tree amplitudes, all with more than three legs, so there must be six negative helicities, while there are only five, three from the triple cut and two from being a MHV amplitude.

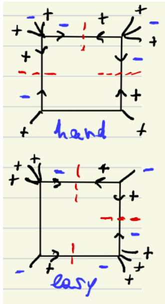


Figure 2.9: Triple cut of hard and easy two-mass boxes of a MHV amplitude.

Five negative helicities suffice to the *easy* two-mass box, for which the massive K are on opposite vertices, because wherever we place the triple cut there is always a tree three-point amplitude, which only needs one negative helicity, so one needs a total of five, see fig. 2.9.

To summarise,

$$d^{4m} = d^{3m} = d^{2m,h} = 0\,, \qquad \text{for MHV amplitudes}\,, \tag{2.47}$$

$$d^{4m} = 0\,, \qquad \text{for NMHV amplitudes}\,. \tag{2.48}$$

By counting negative helicities, one can also show that the amplitudes $A(1^{\pm}, 2^{+}, \ldots, n^{+}$, which vanish at tree level but do not at one loop, have no cuts at all. They are only made of finite, rational terms.

As a particular example, we consider the computation of a one-mass coefficient d^{1m} of the MHV amplitude $A_5^{[1]}(1^- \; 2^- \; 3^+ \; 4^+ \; 5^+)$ (see sec 6.3 of [9]). There are five one-mass boxes, as in fig. 2.10,

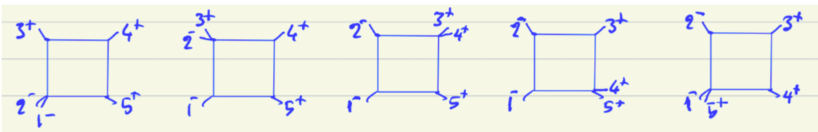


Figure 2.10: One-mass boxes contributing to the MHV amplitude $A_5^{[1]}(1^- \; 2^- \; 3^+ \; 4^+ \; 5^+)$.

where the *massive* leg contains two momenta, $K_{ij} = P_i + P_j$, however, the reflection,

$$A_5^{[1]}(1^- \; 2^- \; 3^+ \; 4^+ \; 5^+) = -A_5^{[1]}(2^- \; 1^- \; 5^+ \; 4^+ \; 3^+)\,, \tag{2.49}$$

relates $I(K_{51})$ to $I(K_{23})$, and $I(K_{45})$ to $I(K_{34})$, and one needs to compute only three mass-boxes, e.g. $I(K_{12})$, $I(K_{23})$ and $I(K_{34})$. As an example, in the Tutorials we compute $I(K_{12})$.

2.3.2 Triple and double cuts

The triple cut is defined by

$$\ell_1^2 = \ell_2^2 = \ell_3^2 = 0\,. \tag{2.50}$$

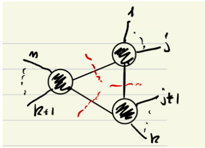


Figure 2.11: Triple cut.

It receives contributions also from box integrals, as we have seen in the example of the two-mass boxes for MHV amplitudes, so the box contributions must be subtracted before the triangle coefficient c_i can be computed. Likewise, the double cut receives contributions from box and triangle integrals, which must be subtracted before the bubble coefficient b_i can be properly identified.

We shall not consider here how the triple and double cuts are computed. More details, and references, can be found in sec. 6.4 in [9].

2.3.3 The rational term

Finally, there is the rational part R_n. There are various ways of computing it. So far, we have considered cut momenta in four dimensions. Because in dimensional regularisation the loop momentum is in $D = 4 - 2\epsilon$ dimensions, one way is to deal with the cut momenta in D dimensions, taking $\epsilon < 0$ and treating the (-2ϵ) dimensions as a fifth dimension. This approach, termed D-dimensional unitarity, needs also a quintuple cut, but not always: the measure has the $\mathrm{d}^{-2\epsilon}\ell$ term, which yields an integral of $\mathcal{O}(\epsilon)$. This can be discarded, unless there are (-2ϵ) components of the numerator, which yield a $1/\epsilon$ term.

Another way of computing R_n is to use the on-shell recursion relation on the (integrated) one-loop amplitude. Schematically, we can write $A_n^{(1)} = C_n + R_n$, where C_n labels the four-dimensional cut-constructible terms. C_n comes from branch cuts, and so it is made of logarithms, dilogarithms and π^2 terms.

Let us consider a complex-z dependent shift on the one-loop amplitude, $A_n^{(1)}(z) = C_n(z) + R_n(z)$. In sec. 1.13, we have seen that the z shift on the tree amplitude yields poles corresponding to the multiparticle factorisation of the amplitude into two lower-point amplitudes. In the one-loop amplitude, the shift on C_n would yield branch cuts in z. If C_n has already been computed, it is best to shift only $R_n = A_n^{(1)} - C_n \to R_n(z)$.

R_n may have physical poles and spurious poles. Since $A_n(z)$ has no spurious poles, $C_n(z)$ and $R_n(z)$ must have spurious poles which cancel each other. Thus, the spurious poles of $R_n(z)$ can be determined from the residues of $C_n(z)$, and can be subtracted. More details are found in the review [41].

Based on the procedure we have outlined to decompose the one-loop n-point amplitude, and a few more variations of it, several automated computer programs for generating one-loop amplitudes have been developed.

In order to evaluate NLO QCD corrections to scattering processes, one needs also the corresponding tree amplitudes with one more parton in the final state, and an efficient integration over the phase space of the additional parton. These methods to evaluate the cross sections at NLO already existed, and put together with the automated programs for generating one-loop amplitudes, they led to a rapid evaluation of many scattering processes at NLO accuracy, which was called the NLO revolution.

2.4 Landau conditions

In $d = 4 - 2\epsilon$ dimensions, a one-loop integral with numerators and higher powers of the propagators can always be reduced to a linear combination of scalar integrals with propagators raised to unit powers. Let us then consider the one-loop scalar integral of the n-point function of momenta $\{p_i\}$ and masses $\{m_i\}$,

$$I_n^{(1)}\big(\{p_i\}\ ;\ \{m_i\}\big) = e^{\gamma_E \epsilon} \int \frac{\mathrm{d}^d \ell}{i\pi^{d/2}} \prod_{j=1}^{n} \frac{1}{(\ell - q_j)^2 - m_j^2 + i\epsilon}\,, \tag{2.51}$$

with

$$\sum_i p_i^\mu = 0\,, \qquad q_1 = 0\,, \quad q_j = \sum_{i=1}^{j-1} p_i\,. \tag{2.52}$$

We introduce the Feynman parametrisation,

$$\prod_{j=1}^{n} \frac{1}{(\ell - q_j)^2 - m_j^2} = (n-1)!\ \prod_{j=1}^{n} \int \mathrm{d}\alpha_j\ \frac{\delta\big(1 - \sum_{i=1}^{n} \alpha_i\big)}{D^n}\,, \tag{2.53}$$

with

$$D = \sum_{i=1}^{n} \alpha_i\ \big((\ell - q_i)^2 - m_i^2\big)\,, \tag{2.54}$$

so

$$I_n^{(1)} = (n-1)!\ \frac{e^{\gamma_E \epsilon}}{i\pi^{d/2}} \prod_{j=1}^{n} \int \mathrm{d}\alpha_j\ \delta\big(1 - \sum_{i=1}^{n} \alpha_i\big) \int \mathrm{d}^d \ell\ D^{-n}\,. \tag{2.55}$$

We must find the positions of poles and branch points of $I_n^{(1)}$ as a function of the external momentum $\{p_i\}$. Singularities arise from zeros of $D(\{x_i\}, \ell, \{p_r\})$, but not all zeros yield a singularity: in the complex plane of ℓ, isolated poles can always be avoided by a contour deformation. Singularities occur when contour deformations cannot avoid a pole.

This can happen in two instances:

1. The pole is at an end-point of a contour of integration.

2. Two poles merge on either side of a contour. This yields a pinch singularity, as depicted in fig. 2.12

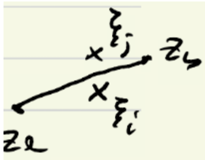


Figure 2.12: Pinch singularity.

The integrand of eq. (2.55) is analytic in ℓ, except at

$$D = \sum_{i=1}^{n} \alpha_i \left((\ell - q_i)^2 - m_i^2 \right) = 0\,. \tag{2.56}$$

D is quadratic in ℓ, thus the contour is trapped if the two roots of $D = 0$ merge. This occurs at

$$\left.\frac{\partial D}{\partial \ell}\right|_{D=0} = 0 \qquad \Rightarrow \qquad \sum_i a_i\, (\ell - q_i) = 0\,. \tag{2.57}$$

Eqs. (2.56) and (2.57) are the Landau equations, a set of necessary conditions to have a pinch singularity.

For $\ell_i^2 \neq m_i^2$, we must have $\alpha_i = 0$. Then we can state that

$$\sum_{i \in C} \alpha_i\, (\ell - q_i) = 0\,, \tag{2.58}$$

where C is the set of c cut propagators, for which

$$(\ell - q_i)^2 - m_i^2 = 0, \qquad i = 1, \ldots, c\,. \tag{2.59}$$

Multiplying (2.58) by $(\ell - q_j)$ we get the matrix equation,

$$\begin{pmatrix} (\ell - q_1)\cdot(\ell - q_1) & \cdots & (\ell - q_1)\cdot(\ell - q_c) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ (\ell - q_c)\cdot(\ell - q_1) & \cdots & (\ell - q_c)\cdot(\ell - q_c) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_c \end{pmatrix} = 0\,. \tag{2.60}$$

The system has non-trivial solutions only if the (Gram) determinant vanishes.

2.5 Feynman integrals and periods

In sec. 2.2, we said that a one-loop amplitude can be decomposed into a linear combination of scalar integrals, with coefficients which we outlined how to determine through unitarity cuts. The scalar integrals are evaluated and expanded as a Laurent series in the dimensional regularisation parameter $\epsilon = 2 - D/2$. For example, the bubble integral with massless propagators yields

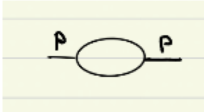


Figure 2.13: Bubble integral with massless propagators.

$$
\begin{aligned}
I_2(p) &= e^{\gamma_E \epsilon} \int \frac{d^D \ell}{i\pi^{D/2}} \frac{1}{\ell^2 (\ell - p)^2} \\
&= \frac{1}{\epsilon} + 2 - \log(-p^2) + \epsilon \Big[\frac{1}{2} \log^2(-p^2) - 2 \log(-p^2) - \frac{\zeta_2}{2} + 4 \Big] + \mathcal{O}(\epsilon^2) \,,
\end{aligned} \tag{2.61}
$$

where

$$
\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{2.62}
$$

is the ζ value, i.e. the Riemann ζ function at integer values of n, which for $n = 1$ diverges. For even values of n, it is given by

$$
\zeta_{2n} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \,, \tag{2.63}
$$

where B_{2n} are the Bernoulli numbers,

$$
B_2 = \frac{1}{6} \,, \quad B_4 = -\frac{1}{30} \,, \quad \ldots \tag{2.64}
$$

so

$$
\zeta_2 = \frac{\pi^2}{6} \,, \quad \zeta_4 = \frac{\pi^4}{90} \,, \quad \ldots \tag{2.65}
$$

Note that the coefficients of the Laurent expansion are real in the Euclidean region, where $p^2 < 0$, and develop a branch cut, starting from $p^2 = 0$, in the Minkowski region.

The triangle integral, with massless propagators and massive external legs, yields

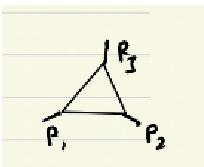


Figure 2.14: Triangle integral.

$$
\begin{aligned}
I_3(p_1,p_2,p_3) &= e^{\gamma_E \epsilon} \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{\ell^2(\ell-p_1)^2(\ell+p_3)^2} \\
&= \frac{2}{\sqrt{\lambda(p_1^2,p_2^2,p_3^2)}} \left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \frac{1}{2}\log(z\bar{z}) \log(\frac{1-z}{1-\bar{z}}) \right] + \mathcal{O}(\epsilon)\,,
\end{aligned} \tag{2.66}
$$

where

$$
\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \tag{2.67}
$$

is the Källen function, and

$$
\frac{p_1^2}{p_2^2} = z\bar{z}\,, \qquad \frac{p_2^2}{p_3^2} = (1-z)(1-\bar{z})\,, \tag{2.68}
$$

and

$$
\log z = \int_1^z \frac{dt}{t}\,, \qquad \mathrm{Li}_n(z) = \int_0^z \frac{dt}{t}\,\mathrm{Li}_{n-1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}\,, \tag{2.69}
$$

are the logarithm and the classical polylogarithm Li_n. In particular, in one-loop integrals we need the dilogarithm,

$$
\mathrm{Li}_2(z) = \int_0^z \frac{dt}{t}\,\mathrm{Li}_1(z)\,, \tag{2.70}
$$

with

$$
\mathrm{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z)\,. \tag{2.71}
$$

Note that

$$
\mathrm{Li}_n(1) = \zeta_n\,, \tag{2.72}
$$

for $n > 1$.

Because of the massive external legs, the Laurent expansion of the triangle integral has no poles in ϵ (further, it cannot have branch cuts, thus it is written in terms of single-valued functions ...).

Logarithms, dilogarithms and ζ values are the entities which usually occur in one-loop amplitudes. Logarithms and dilogarithms are multi-valued functions: they have branch cuts. Further, they may depend on more than one variable.

Beyond one loop, there are several ways of evaluating amplitudes. The most popular procedure is to decompose the amplitude into a set of scalar integrals, which are usually not all independent. Through integration-by-part-identities (IBP), described in app. E and in app. H.49 on the example of Higgs production from gluon fusion, one reduces the scalar integrals to a set of linearly independent ones, called master integrals, which are then evaluated through differential equations, which we will discuss in the next lecture.

The master integrals are Laurent expanded,

$$
I = \sum_{k=k_0}^{\infty} I_k\,\epsilon^k\,, \tag{2.73}
$$

with $k_0 \geq -2|\ell|$ and where $|\ell|$ is the number of loops. The coefficients I_k contain classical polylogarithms, more general polylogarithms, and in some cases also elliptic integrals.

Through experience, we have learned that already at two loops the coefficients I_k may be very complicated. Luckily, in the last ten years our knowledge of the coefficients I_k has greatly improved. That is the topic of this lecture, in which we mostly follow Duhr's 2014 TASI lectures [44].

In I_k we have polylogarithms. May we have trigonometric functions, $\log(\pi)$, the Euler number e, $\log(\log(p^2))$, and so on ? What kind of functions may we actually have ?

In order to answer this question, we review the properties of numbers. The fundamental theorem of algebra states that: Every single-variable degree-n polynomial equation,

$$a_0 + a_1\, z + \ldots + a_n\, z^n = 0\,, \tag{2.74}$$

with rational coefficients $a_i \in \mathbb{Q}$, has n complex roots.

Thus, a complex number is called algebraic, over the field $\mathbb{Q}$ of the rational numbers, if it is the root of a polynomial with rational coefficients. Algebraic numbers form also a field, called $\overline{\mathbb{Q}}$, since sums and products of algebraic numbers are algebraic and the inverse of an algebraic number is algebraic. All the rational numbers are also algebraic: if q is rational, it is also the root of $z - q$, so it is also algebraic, so $\mathbb{Q} \subset \overline{\mathbb{Q}}$. Then, every root $\sqrt[n]{q}$ is algebraic, since it is the root of $z^n - q$. All roots of unity, $z^n - 1$, and in particular i, are algebraic. The inverse of $\sqrt{n}$, with n a natural number, is algebraic, since it is the root of $nz^2 - 1 = 0$.

A complex number that is not algebraic is termed *transcendental*. However, there is a big difference in size between algebraic and transcendental numbers: algebraic numbers are countable (every polynomial has a finite number of roots), while complex numbers, and so transcendental numbers, are not. Thus, it is usually difficult to show that a complex number is transcendental. One can use the theorem of Hermite-Lindemann, which states that if z is a non-zero complex number, then either z or e^z are transcendental. E.g. e is transcendental, because $e = e^1$ and 1 is not transcendental. π is transcendental, because $-1 = e^{i\pi}$ and i are algebraic. Thus, also π^n and ζ_{2n} are transcendental.

With the same definitions, changing numbers with functions, the algebraic and transcendental notions are extended to functions. E.g. $\sqrt{x^2 + y^2}$ is an algebraic function, since it is the root of $z^2 - (x^2 + y^2) = 0$. $\log q$ is transcendental for all algebraic q, since $q = e^{\log q}$ is algebraic.

We have seen that, among the Laurent coefficients of one-loop integrals, ζ_2 and the logarithm are transcendental. Hermite-Lindemann theorem cannot say if the classical polylogarithms or the ζ_n values with odd n are transcendental, but they are believed to be so.

We need another class of numbers, the periods.

A period is a complex number whose real and imaginary parts are values of integrals of algebraic functions with algebraic coefficients, over the domain given by polynomial inequalities with algebraic coefficients. E.g.

- all algebraic numbers are periods, since $\sqrt{n} = \displaystyle\int_{nx^2 \leq 1} dx$;

- π is a period since $\pi = \iint_{x^2+y^2\leq 1} dx\ dy$;
- the logarithm of an algebraic number is a period since $\log q = \int_1^q \frac{dt}{t}$;
- The dilogarithm is a period, since $\text{Li}_2(z) = \int_{0\leq t_2\leq t_1\leq z} \frac{dt_1\ dt_2}{t_1(1-t_2)}$;
- all classical polylogarithms Li_n are periods;
- the ζ_n values, with integer n, are periods;
- The perimeter of an ellipse with radii a and b is the elliptical integral,

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\,dx\,, \tag{2.75}$$

  and it is a period.

However, $e,\ \gamma_E,\ 1/\pi,\ \log\pi$ are not periods.

Periods are countable (because they are defined through algebraic numbers, which are countable), and form a ring $\mathbb{P}$, because sums and products of periods are periods, but the inverse of a period, e.g. π, is not a period. Then one has the inclusion,

$$\mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathbb{P} \subset \mathbb{C}\,. \tag{2.76}$$

A theorem due to Bogner and Weinzierl [45], states that: in the (Euclidean) region, where all Mandelstam invariants s are non-positive, $s \leq 0$, and all masses are non-negative, $m \geq 0$, and where in addition all ratios of invariants and masses are rational, the coefficients of the Laurent expansion of a Feynman integral are periods.

The idea is to use the Feynman parameter representation of the integral, and to show that every term in the ϵ expansion is an integral of a rational function over a rational domain, and thus a period. Note that $e^{\gamma_E\epsilon}$ and $\pi^{D/2}$ were put in the overall normalisation of the Feynman integrals precisely to cancel terms of γ_E and $\log\pi$, which are not periods.

Bogner-Weinzierl theorem answers our original question: numbers and functions which are not periods, like $\log\pi$, trigonometric functions, Euler number e, $\log(\log\pi^2)$, cannot appear in Feynman integrals.

2.6 Multiple polylogarithms

We introduce now a generalisation of the classical polylogarithms, called multiple polylogarithms (MPL). Like the classical polylogairthms, they may be defined through an iterated integral [46, 47],

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\,G(a_2,\ldots,a_n;t)\,, \tag{2.77}$$

with $G(;z) = 1$ and with $z, a_i \in \mathbb{C}$, and where the a_i are not all zero. In the case that $a_1 = \ldots = a_n = 0$, we define

$$G(\overbrace{0, \ldots, 0}^{n}; z) \equiv G(\vec{0}_n; z) = \frac{1}{n!} \log^n(z)\,, \tag{2.78}$$

where $\vec{a} = (a_1, \ldots, a_n)$ is the vector of roots, and its dimension is called the weight of the MPL. For algebraic values of the arguments, the MPLs are periods. In general, it is expected (although not proven) that they are transcendental.

Logarithms and classical polylogarithms are special cases of MPLs,

$$G(\underbrace{a, \ldots, a}_{n}; z) \equiv G(\vec{a}_n; z) = \frac{1}{n!} \log^n\!\left(1 - \frac{z}{a}\right), \tag{2.79}$$

$$G(\underbrace{0, \ldots, 0}_{n-1}, 1; z) \equiv G(\vec{0}_{n-1}, 1; z) = -\mathrm{Li}_n(z)\,. \tag{2.80}$$

for which the roots are all zeros, except for $\mathrm{Li}_1(z)$. Also the harmonic polylogarithms [48] (HPL) defined as

$$H(a_1, \ldots, a_n; z) = \int_0^z dt\; f(a_1; t)\; H(a_2, \ldots, a_n; t)\,, \tag{2.81}$$

with

$$f(1;t) = \frac{1}{1-t}; \qquad f(-1;t) = \frac{1}{1+t}; \qquad f(0;t) = \frac{1}{t}\,, \tag{2.82}$$

are special cases of MPLs, where the roots are equal to $+1,\ 0,\ -1$.

$$H(\vec{a}; z) = (-1)^p\; G(\vec{a}, z)\,, \tag{2.83}$$

where p is the number of roots in $\vec{a}$, which equal $+1$. They appear in many amplitudes at two loops and beyond. Likewise, the two-dimensional harmonic polylogarithms [49] are special cases of MPLs, with $a_i \in \{0, 1, -y, -1, -y\}$. They appeared first in the computation of two-loop four point functions with three massless and one massive leg. So are also the generalised harmonic polylogarithms [50],

$$G(-r, \vec{a}; z) = \int_0^z \frac{dt}{\sqrt{t(4+t)}}\; G(\vec{a}; t)\,, \tag{2.84}$$

which are defined as iterated integrals over a radical,

$$f(-r, t) = \frac{1}{\sqrt{t(4+t)}}\,. \tag{2.85}$$

They can be expressed in terms of MPLs by rationalising the square root via the change of variables,

$$t = \frac{(1-\eta)^2}{\eta}\,, \tag{2.86}$$

then

$$\frac{dt}{\sqrt{t(4+t)}} = -\frac{d\eta}{\eta}\,, \tag{2.87}$$

and eq. (2.84) becomes

$$G(-r,\vec{a};z) = -\int_1^{\xi} \frac{d\eta}{\eta}\, G\big(\vec{a};\frac{(1-\eta)^2}{\eta}\big)\,, \tag{2.88}$$

with $z = \dfrac{(1-\xi)^2}{\xi}$. These integrals usually occur in loop amplitudes with a two-particle threshold at $s = 4m^2$, and where $z = -\dfrac{s}{m^2}$.

Another example of MPLs are the cyclotomic harmonic polylogarithms (see ref. [44] and references therein).

2.6.1 Other definitions of multiple polylogarithms

In the mathematical literature, MPLs are defined in a slightly more general way, with a generic base point a_0,

$$I(a_0;a_1,\ldots,a_n;a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{dt}{t-a_n}\, I(a_0;a_1,\ldots a_{n-1};t)\,, \tag{2.89}$$

with $I(a_0;a_1) = 1$. Then

$$G(a_1,\ldots,a_n;z) = I(0;a_n,\ldots,a_1;z)\,, \tag{2.90}$$

and one can easily express the I's as linear combinations of integrals at base point zero, e.g.

$$I(a_0;a_1;a_2) = \int_{a_0}^{a_2} \frac{dt}{t-a_1}\, I(a_0;t) = \Big(\int_0^{a_2} - \int_0^{a_0}\Big)\frac{dt}{t-a_1} = G(a_1;a_2) - G(a_1;a_0)\,, \tag{2.91}$$

In app. H.52, the relation between I's and G's is provided for weight-2 and weight-3 MPLs.

Just like the classical polylogarithms, also the MPLs can be defined through a power series,

$$\mathrm{Li}_{m_1,\ldots,m_k}(z_1,\ldots,z_k) = \sum_{0<n_1<n_2<\ldots<n_k} \frac{z_1^{n_1} z_2^{n_2}\cdots z_k^{n_k}}{n_1^{m_1} n_2^{m_2}\cdots n_k^{m_k}}\,, \tag{2.92}$$

where $|z_i| < 1$, for the sums to converge. The Li's are related to the G's by

$$\mathrm{Li}_{m_1,\ldots,m_k}(z_1,\ldots,z_k) = (-1)^k G(\underbrace{0,\ldots,0}_{m_k-1},\frac{1}{z_k},\ldots,\underbrace{0,\ldots,0}_{m_1-1},\frac{1}{z_1\cdots z_k};1)\,, \tag{2.93}$$

where the number k of indices is called the depth of the MPL.

One can show that up to weight three all MPLs can be expressed in terms of classical polylogarithms. At weight four, non-classical functions, like $\mathrm{Li}_{2,2}$, start appearing.

2.6.2 Properties of multiple polylogarithms

From the integral definition of MPLs (2.77), we see that $G(a_1, \ldots, a_n; z)$ is analytic at $z = 0$ when $a_n \neq 0$. In particular,

$$\lim_{z \to 0} G(a_1, \ldots, a_n; z) = 0\,, \tag{2.94}$$

when $a_n \neq 0$, i.e. around $z = 0$, $G(a_1, \ldots, a_n; z)$ admits a Taylor expansion in z, with a null constant term. Further, $G(a_1, \ldots, a_n; z)$ is divergent at $z = a_1$.

MPLs are multi-valued functions, with branch cuts that may be extend from any a_i to ∞, e.g.

- $G(\vec{a}_n; z) = \frac{1}{n!} \log^n\big(1 - \frac{z}{a}\big)$ has a branch cut from $z = a$ to $z = \infty$;
- $G(0, 1; z) = -\mathrm{Li}_2(z)$ has a branch cut from $z = 1$ to $z = \infty$, but no cuts starting from 0.

If $a_n \neq 0$, $G(\vec{a}_n; z)$ is invariant under a rescaling of the arguments,

$$G(k\vec{a}; kz) = G(\vec{a}; z)\,, \quad \forall k \in \mathbb{C}^*\,. \tag{2.95}$$

Further, if $a_1 \neq 1$ and $a_n \neq 0$, Hölder identity,

$$G(a_1, \ldots, a_n; 1) = \sum_{k=0}^{n} (-1)^k G\big(1 - a_k, \ldots, 1 - a_1; 1 - \frac{1}{z}\big) G\big(a_{k+1}, \ldots, a_n; \frac{1}{z}\big)\,, \tag{2.96}$$

holds $\forall z \in \mathbb{C}^*$. If $z \to \infty$, the second MPL vanishes, unless it has no a_i's, i.e. for $k = n$, and the identity becomes

$$G(a_1, \ldots, a_n; 1) = (-1)^n G(1 - a_n, \ldots, 1 - a_1; 1)\,. \tag{2.97}$$

2.6.3 The shuffle algebra

The product of two MPLs of weight one,

$$G(a; z)\ G(b; z) = \int_0^z \frac{dt_1}{t_1 - a} \int_0^z \frac{dt_2}{t_2 - b}\,, \tag{2.98}$$

can be written as an integral over the square with corners $(0,0),\ (0,z),\ (z,0),\ (z,z)$,

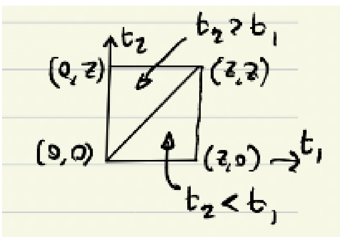


Figure 2.15: Integration domain of eq. (2.98).

which can be written as a sum over two triangles,

$$G(a;z)\ G(b;z) = \int_0^z \frac{dt_1}{t_1-a}\int_0^{t_1}\frac{dt_2}{t_2-b} + \int_0^z \frac{dt_2}{t_2-b}\int_0^{t_2}\frac{dt_1}{t_1-a} = G(a,b;z)+G(b,a;z)\,, \tag{2.99}$$

i.e. the product of two MPLs of weight one yields a linear combination of MPLs of weight two.

Likewise, the product of two MPLs of higher weight can be written as an integral over a hypercube, and then split along the diagonals into iterated integrals. In fact, the product of two MPLs of weights n_1 and n_2 can be written as linear combination of MPLs of weight n_1+n_2,

$$G(a_1,\ldots,a_{n_1};z)\ G(a_{n_1+1},\ldots,a_{n_1+n_2};z) = \sum_{\sigma\in\{n_1\}\sqcup\!\sqcup\{n_2\}} G(a_{\sigma_1},\ldots,a_{\sigma_{n_1+n_2}};z)\,, \tag{2.100}$$

where the sum is over all the shuffles $\{n_1\}\sqcup\!\sqcup\{n_2\}$ of n_1+n_2 elements, the shuffles being the permutations which preserve the ordering of the a_i within $(a_1,\ldots,a_{n_1})$ and of the a_j within $(a_{n_1+1},\ldots,a_{n_1+n_2})$, while allowing for all possible orderings of the a_i with respect to the a_j. The number of shuffles is given by the binomial coefficient $\begin{pmatrix} n_1+n_2 \\ n_1 \end{pmatrix} = \frac{(n_1+n_2)!}{n_1!\ n_2!}$. We already introduced the shuffle in the context of the Kleiss-Kuijff relations, sec. 1.7.1. E.g.

$$\begin{aligned} G(a,b;z)\ G(c;z) &= G(a,b,c;z)+G(a,c,b;z)+G(c,a,b;z)\,, && (2.101)\\ G(a,b;z)\ G(c,d;z) &= G(a,b,c,d;z)+G(a,c,b,d;z)+G(c,a,b,d;z) \\ &+ G(a,c,d,b;z)+G(c,a,d,b;z)+G(c,d,a,b;z)\,. && (2.102) \end{aligned}$$

The MPLs form a shuffle algebra, i.e. a vector space equipped with a (shuffle) product. The algebra is graded, because the shuffle product preserves the weight.

Since we know that $G(a_1,\ldots,a_n;z)$ is analytic at $z=0$, when $a_n\neq 0$, we can use the shuffle algebra in order to have MPLs with non-zero rightmost index, except for $G(\vec{0}_n,z)$, e.g.

$$G(b,0,0;z) = G(b;z)\ G(0,0;z) - G(0,b,0;z) - G(0,0,b;z)\,, \tag{2.103}$$

then

$$G(0,b,0;z) = G(0,b;z)\ G(0;z) - 2\ G(0,0,b;z) \tag{2.104}$$

so

$$\begin{aligned} G(b,0,0;z) &= G(0,0,b;z) - G(0,b;z)\ G(0;z) + G(b;z)\ G(0,0;z) \\ &= -\mathrm{Li}_3\big(\frac{z}{b}\big) + \mathrm{Li}_2\big(\frac{z}{b}\big)\ \log(z) + \log\big(1-\frac{z}{b}\big)\ \frac{1}{2}\ \log^2(z)\,. && (2.105) \end{aligned}$$

Using the sum definition of the MPLs (2.92), one can see that they also form a stuffle algebra. Just like the shuffle, the stuffle product preserves the weight, but not the depth. Examples of stuffle products of MPLs as nested sums can be found in Duhr's TASI lectures [44].

2.6.4 Multiple zeta values

We saw that the classical polylogarithms at $z = 1$ yield $\mathrm{Li}_n(1) = \zeta_n$, eq. (2.72), i.e. the ζ values, which are periods and (likely) transcendental. Given $m_1, \ldots, m_k$ positive integers, we define the multiple zeta values (MZV),

$$\zeta_{m_1,\ldots,m_k} = \mathrm{Li}_{m_k,\ldots,m_1}(1, \ldots, 1) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1^{m_1} \ldots n_k^{m_k}} \,. \tag{2.106}$$

For $m_1 = 1$, $\zeta_{m_1,\ldots,m_k}$ is divergent.

Weight and depth of a MZV are defined as for a MPL. Just like the Li's, also the MZVs admit an integral representation in terms of the G's, which implies that also the MZV's are periods. Also MZVs are common in multi-loop computations. Details about the relations among MZVs are found in Duhr's TASI lectures [44].

2.6.5 Hopf algebra of MPLs

$$\log(a\ b) = \log(a) + \log(b) \tag{2.107}$$

is the functional equation among logarithms. From this, all relations among logarithms can be found. Tables of logarithms, based on numerical values of logarithms and endowed with the functional equation (2.107), have been used for centuries. Examples of functional equations among dilogarithms are

$$\mathrm{Li}_2(1-z) \;=\; -\mathrm{Li}_2(z) - \log z\, \log(1-z) + \zeta_2 \,, \tag{2.108}$$

$$\mathrm{Li}_2\big(1-\frac{1}{z}\big) \;=\; -\mathrm{Li}_2(1-z) - \frac{1}{2}\, \log^2 z \,. \tag{2.109}$$

Eq. (2.108) will be proven in app. H.57. More identities can be found on Lewin's book on polylogarithms [51]. The number of functional equations grows rapidly with the weight, and the shuffle and stuffle relations are not enough to account for all of them.

Since MPLs are the norm in multi-loop computations, functional equations among them are a must have. Not only are they essential to simplify an analytic computation, or to help in analytically continuing the MPLs, they are useful also when we are only interested in a numerical answer out of an analytic computation, because they allow one to minimise the number of MPLs for which it is necessary to run a numerical routine.

Now, we introduce an algebraic method that allows one to derive functional equations among MPLs. Firstly, we define A_n as the vector space of MPLs of weight n, and the vector space of all MPLs as

$$A = \bigoplus_{n=0}^{\infty} A_n, \quad A_0 = \mathbb{Q} \,. \tag{2.110}$$

Of course, the definition as a direct sum makes sense only if there are no relations among MPLs of different weights.

We already know that A is an algebra with respect to both the shuffle and the stuffle products. In general, we can view the product as a map μ from $A \otimes A$ to A, which assigns to a pair of elements $a \otimes b$ their product $a \cdot b$. We know that the product is associative,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)\,, \tag{2.111}$$

and distributive,

$$\begin{aligned} (a + b) \cdot c &= a \cdot c + b \cdot c\,, \\ a \cdot (b + c) &= a \cdot b + a \cdot c\,. \end{aligned} \tag{2.112}$$

Further, also $A \otimes A$ is an algebra, so one can define the product by components,

$$(a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d)\,. \tag{2.113}$$

In addition, one can introduce a coalgebra, i.e. a vector equipped with a coproduct, i.e. a linear map $\Delta :\ A \to A \otimes A$ which assigns to every element $a \in A$ its coproduct $\Delta(a) \in A \otimes A$.

The coproduct must be coassociative,

$$(\Delta \otimes \mathrm{id}) \cdot \Delta = (\mathrm{id} \otimes \Delta) \cdot \Delta\,, \tag{2.114}$$

such that if $\Delta(a) = a_1 \otimes a_2$, then

$$(\Delta \otimes \mathrm{id}) \cdot \Delta(a) = (\Delta \otimes \mathrm{id}) \cdot (a_1 \otimes a_2) = \Delta(a_1) \otimes a_2 = a_{1,1} \otimes a_{1,2} \otimes a_2\,, \tag{2.115}$$

$$(\mathrm{id} \otimes \Delta) \cdot \Delta(a) = (\mathrm{id} \otimes \Delta) \cdot (a_1 \otimes a_2) = a_1 \otimes \Delta(a_2) = a_1 \otimes a_{2,1} \otimes a_{2,2}\,, \tag{2.116}$$

and the two expressions must be the same, i.e. the order in which we iterate the coproduct is irrelevant, so there is a unique way of splitting an element into three or more elements.

If A is equipped with both a product and a coproduct, such that $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$ then we have a bialgebra. If the bialgebra is graded, also the coproduct must preserve the weight, i.e. the sum of the weights of the two factors of $\Delta(a)$ must equal the weight of a.

A Hopf algebra is a bialgebra with an antipode $S :\ A \to A$, such that

$$\begin{cases} S(a \cdot b) = S(b) \cdot S(a)\,, \\ \mu(\mathrm{id} \otimes S)\ \Delta = \mu(S \otimes \mathrm{id})\ \Delta = 0\,. \end{cases} \tag{2.117}$$

We will not use the antipode in what follows, i.e. we will not make distinctions between a Hopf algebra and a bialgebra. An element of a Hopf algebra is primitive if

$$\Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1}\,, \tag{2.118}$$

i.e. it admits only the trivial decomposition. The reduced coproduct is defined as

$$\Delta^{'}(x) = \Delta(x) - (\mathbb{1} \otimes x + x \otimes \mathbb{1})\,, \tag{2.119}$$

so for a primitive element x, $\Delta^{'}(x) = 0$. The coproduct element $\Delta_{i_1,\dots,i_k}$ is defined as the part of the iterated coproduct with weights $(i_1, \dots, i_k)$.

For example, we have a set of letters $\{a, b, c\}$ and a vector space A spanned by linear combinations of words. A is graded, and the weight is given by the length of the word. We define the coproduct as

$$\Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1}, \quad \text{for } x = a, b, c\,, \tag{2.120}$$

i.e. by definition all the letters are primitive. Then

$$\begin{aligned} \Delta(a \cdot b) &= \Delta(a) \cdot \Delta(b) = (\mathbb{1} \otimes a + a \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes b + b \otimes \mathbb{1}) \\ &= \mathbb{1} \otimes (a \cdot b) + (a \cdot b) \otimes \mathbb{1} + a \otimes b + b \otimes a\,, \end{aligned} \tag{2.121}$$

$$\begin{aligned} \Delta(abc) &= \Delta(a) \cdot \Delta(b \cdot c) \\ &= (\mathbb{1} \otimes a + a \otimes \mathbb{1}) \cdot \big(\mathbb{1} \otimes (b \cdot c) + (b \cdot c) \otimes \mathbb{1} + b \otimes c + c \otimes b\big) \\ &= \mathbb{1} \otimes (abc) + (bc) \otimes a + b \otimes (ac) + c \otimes (ab) \\ &+ a \otimes (bc) + (abc) \otimes \mathbb{1} + (ab) \otimes c + (ac) \otimes b\,. \end{aligned} \tag{2.122}$$

It is straightforward to check that the product is associative,

$$\Delta(a) \cdot \Delta(b \cdot c) = \Delta(a \cdot b) \cdot \Delta(c)\,. \tag{2.123}$$

A bit longer, but equally straightforward is to check that the coproduct is associative, i.e.

$$(\Delta \otimes \mathrm{id}) \cdot \Delta(abc) = (\mathrm{id} \otimes \Delta) \cdot \Delta(abc)\,. \tag{2.124}$$

The reduced coproducts are

$$\begin{aligned} \Delta^{'}(a \cdot b) &= a \otimes b + b \otimes a\,, \\ \Delta^{'}(a \cdot b \cdot c) &= a \otimes (b\ c) + b \otimes (a\ c) + c \otimes (a\ b) + (a\ b) \otimes c + (a\ c) \otimes b + (b\ c) \otimes a\,. \end{aligned} \tag{2.125}$$

The coproduct elements are

$$\begin{aligned} \Delta_{1,1}(a \cdot b) &= a \otimes b + b \otimes a\,, \\ \Delta_{2,1}(a \cdot b \cdot c) &= (a\ b) \otimes c + (a\ c) \otimes b + (b\ c) \otimes a\,, \\ \Delta_{1,2}(a \cdot b \cdot c) &= a \otimes (b\ c) + b \otimes (a\ c) + c \otimes (a\ b)\,, \\ \Delta_{1,1,1}(a \cdot b \cdot c) &= a \otimes b \otimes c + b \otimes a \otimes c + b \otimes c \otimes a + a \otimes c \otimes b + c \otimes a \otimes b + c \otimes b \otimes a\,. \end{aligned} \tag{2.126}$$

The MPLs form a Hopf algebra. In order to show it, we use the I form of the MPLs (2.89), and we define the coproduct as

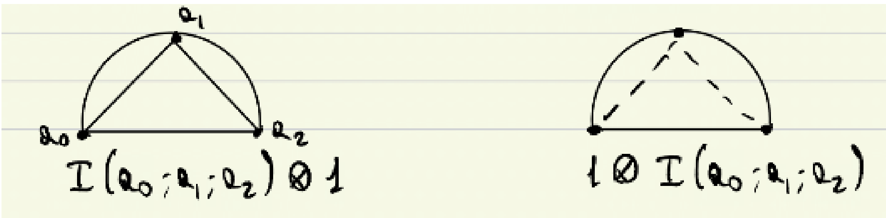
$$\Delta\big(I(a_0; a_1, \ldots, a_n; a_{n+1})\big) = \sum_{0=i_1<i_2<\ldots<i_k<i_{k+1}=n} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \Big[\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}\Big], \quad (2.127)$$

where the a_i are generic, i.e. $a_1 \neq a_2 \ldots \neq a_n \neq a_{n+1} \neq 0$.

The various terms in the sum (2.127) can be generated through a graphic procedure [47, 52]:

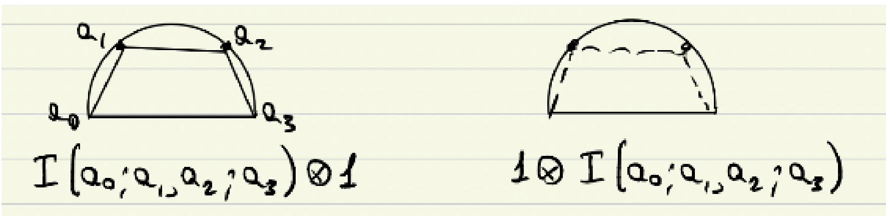
- Draw a semicircle on which $a_0, a_1, \ldots, a_{n+1}$ are distributed clockwise, such that a_0 and a_{n+1} are two end-points.
- Mark some points, $a_{i_1}, \ldots, a_{i_k}$ and draw the convex polygon with vertices $a_0, a_{i_1}, \ldots, a_{i_k}, a_{n+1}$. This polygon defines the first factor in the sum.
- The unmarked points define a set of complementary convex polygons. These polygons define the product in the second factor.

E.g. let us consider the coproduct of a generic MPL of weight one, $\Delta\big(I(a_0; a_1; a_2)\big)$. On the semicircle, we can draw polygons as follows

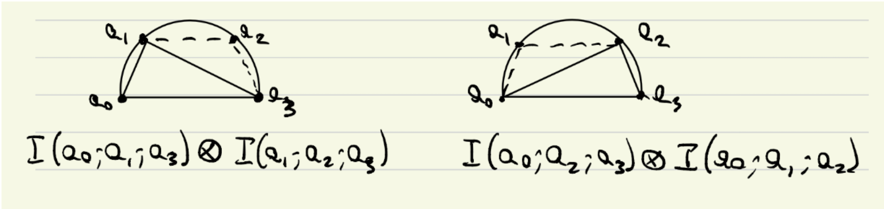


There is only one way to draw a polygon, so the complementary polygon is 1, and we get $I(a_0; a_1; a_2) \otimes 1$. Else, we draw no polygons, then the complementary polygon is $I(a_0; a_1; a_2)$ and we get $1 \otimes I(a_0; a_1; a_2)$: MPLs of weight on are primitive.

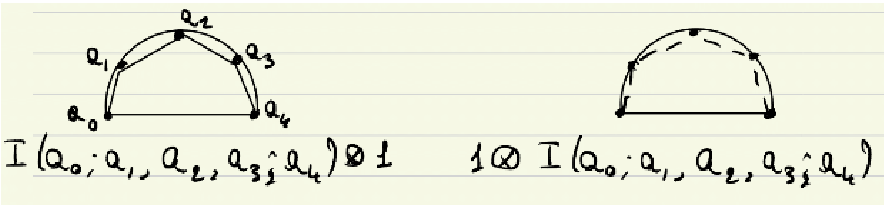
Let us consider the coproduct of a generic MPL of weight two, $\Delta\big(I(a_0; a_1, a_2; a_3)\big)$. We get



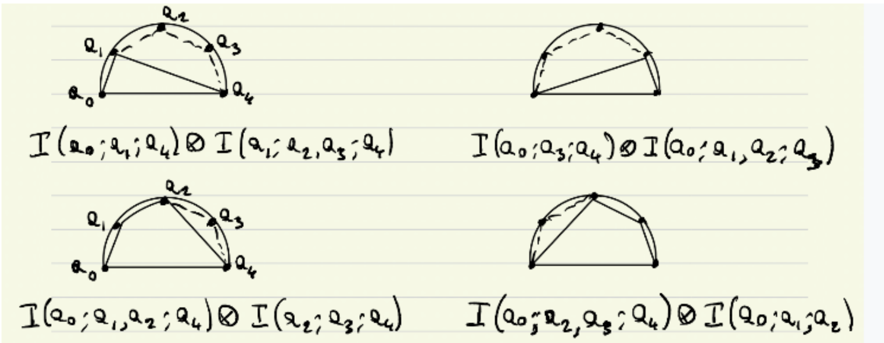
These represent the two trivial terms, however we also have:

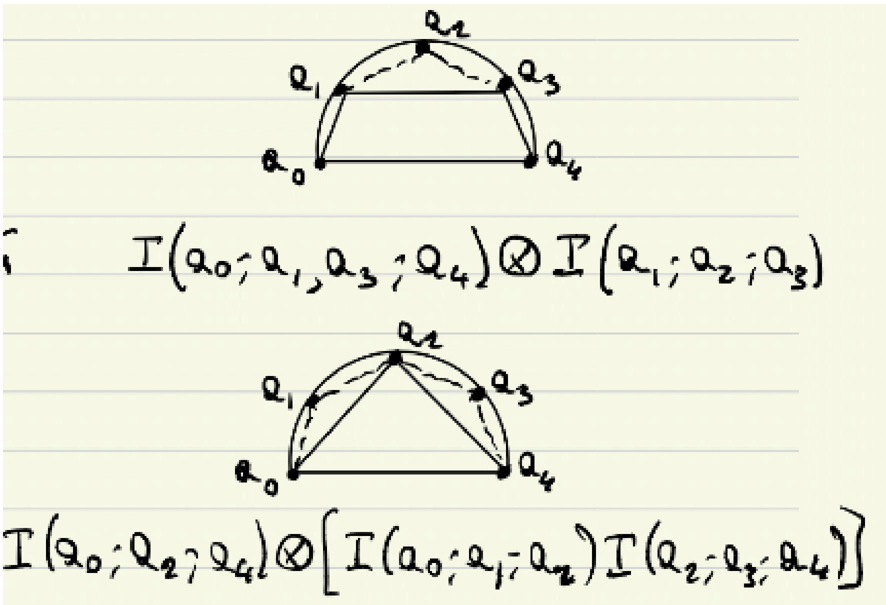


At weight three, $\Delta\big(I(a_0; a_1, a_2, a_3; a_4)\big)$, we get



i.e. the trivial terms, and further





so in addition we also have a term with two complementary polygons.

So far, we have introduced the coproduct (2.127) of the MPL defined as in eq. (2.89), and gave examples up to weight three through a graphical procedure. Using the relation between I and G

versions of the MPL (2.90), one can then obtain the coproduct $\Delta\big(G(a_1,\ldots,a_n;z)\big)$ as long as the roots a_i are generic.

E.g. at weight two, we found that

$$\begin{aligned}\Delta\big(I(0;a_2,a_1;z)\big) &= I(0;a_2,a_1;z)\otimes\mathbb{1}+\mathbb{1}\otimes I(0;a_2,a_1;z)\\ &+ I(0;a_2;z)\otimes I(a_2;a_1;z)+I(0;a_1;z)\otimes I(0;a_2;a_1)\,,\end{aligned}\tag{2.128}$$

which implies that

$$\begin{aligned}\Delta\big(G(a_1,a_2;z)\big) &= G(a_1,a_2;z)\otimes\mathbb{1}+\mathbb{1}\otimes G(a_1,a_2;z)\\ &+ G(a_2;z)\otimes\big[G(a_1;z)-G(a_1;a_2)\big]+G(a_1;z)\otimes G(a_2;a_1)\,,\end{aligned}\tag{2.129}$$

since $I(a_0;a_1;a_2)=G(a_1;a_2)-G(a_1;a_0)$, eq. (2.91).

In particular,

$$\begin{aligned}\Delta\big(G(0,1;z)\big) &= G(0,1;z)\otimes\mathbb{1}+\mathbb{1}\otimes G(0,1;z)\\ &+ G(1;z)\otimes\big[G(0;z)-\cancelto{0}{G(0;1)}\big]+G(0;z)\otimes\cancelto{0}{G(1;0)}\,.\end{aligned}\tag{2.130}$$

Since $G(0,1;z)=-\mathrm{Li}_2(z)$, we obtain

$$\Delta\big(\mathrm{Li}_2(z)\big)=\mathrm{Li}_2(z)\otimes\mathbb{1}+\mathbb{1}\otimes\mathrm{Li}_2(z)+\mathrm{Li}_1(z)\otimes\log(z)\,.\tag{2.131}$$

At weight three, neglecting the two trivial terms we found that

$$\begin{aligned}\Delta'\big(I(0;a_1,a_2,a_3;z)\big) &= I(0;a_1;z)\otimes I(a_1;a_2,a_3;z)+I(0;a_3;z)\otimes I(0;a_1,a_2;a_3)\\ &+ I(0;a_1,a_2;z)\otimes I(a_2,a_3;z)+I(0;a_2,a_3;z)\otimes I(0;a_1;a_2)\\ &+ I(0;a_1,a_3;z)\otimes I(a_1;a_2,a_3)\\ &+ I(0;a_2;z)\otimes\big[I(0;a_1;a_2)I(a_2;a_3;z)\big]\,,\end{aligned}\tag{2.132}$$

which allows one to obtain the coproduct $\Delta\big(G(a_1,a_2,a_3;z)\big)$ at generic values of a_1,a_2,a_3. Let us see what happens if we specify eq. (2.132) to the reduced coproduct of

$$I(0;1,0,0;z)=G(0,0,1;z)=-\mathrm{Li}_3(z)\,.\tag{2.133}$$

We obtain

$$\begin{aligned}\Delta'\big(I(0;1,0,0;z)\big) &= I(0;1;z)\otimes I(1;0,0;z)+I(0;0;z)\otimes I(0;1,0;0)\\ &+ I(0;1,0;z)\otimes I(0,0;z)+I(0;0,0;z)\otimes I(0;1;0)\\ &+ I(0;1,0;z)\otimes I(1;0,0)+I(0;0;z)\otimes\big[I(0;1;0)I(0;0;z)\big]\,.\end{aligned}\tag{2.134}$$

We need to convert the $I's$ into $G's$, in particular $I(a_1; a_2, a_3; z)$ (see Tutorial), so we get

$$
\begin{aligned}
I(1;0,0;z) &= G(0,0;z) - \cancelto{0}{G(0,0;1)} - \cancelto{0}{G(0;1)}\left[G(0;z) - \cancelto{0}{G(0;1)}\right] = \frac{1}{2}\log^2(z)\,, \\
I(0;1,0;z) &= G(0,1;z) = -\mathrm{Li}_2(z)\,, \\
I(0;1,0,0) &= -\mathrm{Li}_2(0) = 0\,, \\
I(0;1;0) &= G(1;0) = \log 1 = 0\,.
\end{aligned}
\tag{2.135}
$$

However,

$$
\begin{aligned}
I(0;0;z) &= G(0;z) - G(0;0)\,, \\
I(1;0;0) &= G(0;0) - \cancelto{0}{G(0;1)},
\end{aligned}
\tag{2.136}
$$

where $G(0;0)$ is divergent, since $z = a_1$. So as it stands eq. (2.134) is ill-defined, and we cannot use it to compute the coproduct of $\mathrm{Li}_3(z)$.

2.6.6 Shuffle regularisation

The procedure to determine the coproduct of the MPLs that we displayed in sec. 2.6.5 is valid only for generic values of the a_i's. In order to be able to apply it for all values of the a_i's, we must first (shuffle) regularise all the MPLs. We do it as follows:

1. except for MPLs of the type $G(z,\ldots,z;z)$, we use the shuffle algebra in order to express the divergent MPLs in terms of regular ones.

2. We set $G^{reg}(z,\ldots,z;z) = 0$.

E.g. let us take $G(z,a;z)$, with $a \neq z$, which is divergent since $a_1 = z$. Through the shuffle, we write it as

$$
G(z,a;z) = G(z;z)\ G(a;z) - G(a,z;z)\,,
\tag{2.137}
$$

then $G^{reg}(z,a;z) = -G(a,z;z)$ (see also app. H.53 for an example of a weight-3 MPL).

Note that we have already implicitly been using the shuffle regularisation. We fixed $G(0;z) = \log z$ but according to the definition of MPL, $G(0;z)$ should be $\int_0^z \frac{dt}{t}$, which is divergent. In fact,

$$
\log z = \int_1^z \frac{dt}{t} = I(1;0;z) = \int_0^z \frac{dt}{t} - \int_0^1 \frac{dt}{t}\,.
\tag{2.138}
$$

So

$$
G(0;z) = \int_0^z \frac{dt}{t}, \qquad \text{before the regularisation}\,.
\tag{2.139}
$$

$$
G^{reg}(0;z) = \int_0^z \frac{dt}{t} - \int_0^1 \frac{dt}{t} = \log z\,, \qquad \text{after the regularisation}\,.
\tag{2.140}
$$

Further, the regularised product of two MPLs equals the product of the two regularised MPLs, i.e.

the regularisation preserves the multiplication (see app. H.54),

$$\left[G(\vec{a};z)\; G(\vec{b};z)\right]^{reg} = G^{reg}(\vec{a};z)\, G^{reg}(\vec{b};z)\,. \tag{2.141}$$

In the general case, the coproduct is then defined replacing everywhere the unregularised MPLs with the regularised ones. Of course, when the MPLs converge, the two definitions of MPLs coincide.

Returning to the example of eq. (2.134),

$$\begin{aligned} I^{reg}(0;0;z) &= G(0;z)\,, \\ I^{reg}(1;0;0) &= 0\,, \end{aligned} \tag{2.142}$$

and we can finally write

$$\Delta'\big(I(0;1,0,0;z)\big) = \log(1-z)\otimes \frac{1}{2}\,\log^2 z - \mathrm{Li}_2(z)\otimes \log z\,. \tag{2.143}$$

Thus, re-adding the trivial terms, we have

$$\Delta\left(\mathrm{Li}_3(z)\right) = \mathrm{Li}_3(z)\otimes \mathbb{1} + \mathbb{1}\otimes \mathrm{Li}_3(z) + \mathrm{Li}_1(z)\otimes \frac{1}{2}\,\log^2 z + \mathrm{Li}_2(z)\otimes \log z\,. \tag{2.144}$$

The generalisation to the weight-n classical polylogarithm is

$$\Delta\left(\mathrm{Li}_n(z)\right) = \mathbb{1}\otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \mathrm{Li}_{n-k}(z)\otimes \frac{\log^k z}{k!}\,. \tag{2.145}$$

2.6.7 Coaction

Since MZVs are obtained by setting $z=1$ in the MPLs, we know how to compute the coproduct of MZVs. E.g. setting $z=1$ in $\Delta\left(\mathrm{Li}_n(z)\right)$ we obtain $\Delta\left(\zeta_n\right) = \mathbb{1}\otimes\zeta_n + \zeta_n\otimes 1$. However, this is a problem for even-n ζ values: for example, let us consider $\zeta_4 = \frac{\pi^4}{90} = \frac{2}{5}\zeta_2^2$. So

$$\begin{aligned} \Delta\left(\zeta_4\right) &= \frac{2}{5}\,\Delta\left(\zeta_2\right)^2 = \frac{2}{5}\,\left(\mathbb{1}\otimes\zeta_2 + \zeta_2\otimes\mathbb{1}\right)^2 \\ &= \frac{2}{5}\,\left(\mathbb{1}\otimes\zeta^2 + \zeta^2\otimes\mathbb{1} + 2\zeta_2\otimes\zeta_2\right) = \mathbb{1}\otimes\zeta_4 + \zeta_4\otimes\mathbb{1} + \frac{4}{5}\,\zeta_2\otimes\zeta_2\,, \end{aligned} \tag{2.146}$$

which is at odds with the previous formula for $n=4$. We wil see later that $\Delta(i\pi) = i\pi\otimes\mathbb{1} + \mathbb{1}\otimes i\pi$ would lead to similar problems. The solution proposed by Brown [53] is to introduce a ring of polynomials in $i\pi$ with rational coefficients, so that the algebra A we have been dealing with in sec. 2.6.5 is not the Hopf algebra $\mathcal{H}$ of the MPLs, but

$$A = \mathbb{Q}(i\pi)\otimes\mathcal{H}\,, \tag{2.147}$$

i.e. we remove all powers of $i\pi$ in $\mathcal{H}$, but we allow the coefficients of elements of $\mathcal{H}$ to be also polynomials in $i\pi$ rather than just rational functions. Further, we define the coaction $\Delta: A \to A \otimes \mathcal{H}$, such that

$$\Delta(i\pi) = i\pi \otimes \mathbb{1} \tag{2.148}$$

while Δ coincides with the coproduct defined in sec. 2.6.5 on elements of $\mathcal{H}$. Accordingly,

$$\Delta(\zeta_4) = \zeta_4 \otimes 1 = \frac{2}{5}\,\zeta_2 \otimes \mathbb{1} = \frac{2}{5}\,\Delta(\zeta_2^2)\,, \tag{2.149}$$

has no contradictions anymore.

2.6.8 Derivatives and discontinuities

Derivatives act on the last entry of the coaction,

$$\Delta\big(\frac{\partial}{\partial z}\,G\big) = \big(\mathrm{id} \otimes \frac{\partial}{\partial z}\big)\,\Delta(G)\,. \tag{2.150}$$

Let us verify on $\mathrm{Li}_2(z)$ that eq. (2.150) is correct.

$$\mathrm{Li}_2(z) = \int_0^z \frac{dt}{t}\,\mathrm{Li}_1(t) \quad \Rightarrow \quad \frac{\partial}{\partial z}\mathrm{Li}_2(z) = \frac{\mathrm{Li}_1(z)}{z}\,, \tag{2.151}$$

so the left-hand side of eq. (2.150) yields

$$\Delta\big(\frac{\partial}{\partial z}\,\mathrm{Li}_2(z)\big) = \Delta\big(\frac{\mathrm{Li}_1(z)}{z}\big) = \frac{1}{z}\,\left(\mathrm{Li}_1(z) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{Li}_1(z)\right)\,, \tag{2.152}$$

while the right-hand side of (2.150) yields

$$\begin{aligned}
\big(\mathrm{id} \otimes \frac{\partial}{\partial z}\big)\Delta\big(\mathrm{Li}_2(z)\big) &= \big(\mathrm{id} \otimes \frac{\partial}{\partial z}\big)\big(\mathrm{Li}_2(z) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{Li}_2(z) + \mathrm{Li}_1(z) \otimes \log z\big) \\
&= \mathrm{Li}_2(z) \otimes \cancelto{0}{\frac{\partial}{\partial z}}\mathbb{1} + \mathbb{1} \otimes \frac{\partial}{\partial z}\mathrm{Li}_2(z) + \mathrm{Li}_1(z) \otimes \frac{\partial}{\partial z}\log z \\
&= \mathbb{1} \otimes \frac{\mathrm{Li}_1(z)}{z} + \mathrm{Li}_1(z) \otimes \frac{1}{z} \\
&= \frac{1}{z}(\mathbb{1} \otimes \mathrm{Li}_1(z) + \mathrm{Li}_1(z) \otimes \mathbb{1})\,,
\end{aligned} \tag{2.153}$$

which agrees with eq. (2.152).

Eq. (2.150) provides a way of computing derivatives of MPLs. In fact, if G is a MPL of weight n,

$$\frac{\partial}{\partial z}\,G = \mu\,\big(\mathrm{id} \otimes \frac{\partial}{\partial z}\big)\Delta_{n-1,1}(G)\,, \tag{2.154}$$

with $\mu(a \otimes b) = a \cdot b$.

In app. H.55, we compute $\frac{\partial}{\partial y}\,G(1, 1+y; z)$ using eq. (2.154).

The discontinuity acts on the first entry of the coaction,

$$\Delta(\text{Disc } G) = (\text{Disc} \otimes \text{id})\ \Delta(G)\,. \tag{2.155}$$

Let us see how it works on $G(0; z) = \log z$. The logarithm has a branch cut from $z = 0$ to $z \to \infty$. We can write it as $\log(x + i\epsilon) = \log|x| + i\pi\Theta(-x)$. Then $\text{Disc}(\log x) = 2i\ \text{Im}(\log x) = 2i\pi$. On the right-hand side of eq. (2.155), we have

$$(\text{Disc} \otimes \text{id})\ \Delta(\log z) = (\text{Disc} \otimes \text{id})\ (\log z \otimes \mathbb{1} + \mathbb{1} \otimes \log z) = 2i\pi \otimes \mathbb{1}\,, \tag{2.156}$$

which is consistent with the left-hand side of eq. (2.155),

$$\Delta\big(\text{Disc}(\log z)\big) = \Delta(2i\pi) = 2i\pi \otimes \mathbb{1}\,. \tag{2.157}$$

Note that the coproduct $\Delta(i\pi) = i\pi \otimes \mathbb{1} + \mathbb{1} \otimes i\pi$ would have created a contradiction, as mentioned earlier.

As a further example, in app. H.56, we verify the validity of eq. (2.155) on the dilogarithm, $\text{Li}_2(z)$, which has a branch cut from $z = 1$ to $z \to \infty$, and whose discontinuity is

$$\text{Disc}(\text{Li}_2(z)) = 2i\ \text{Im}(\text{Li}_2(z)) = 2\pi i\ \log z\,. \tag{2.158}$$

We can use eq. (2.155) to compute the discontinuity of a MPL,

$$\text{Disc}(G) = \mu\ (\text{Disc} \otimes \text{id})\ \Delta_{1,n-1}\ G\,, \tag{2.159}$$

with $\mu(a \otimes b) = a \cdot b$, e.g. the coproduct element $\Delta_{1,1}$ on $\text{Li}_2(z)$ is $\Delta_{1,1}\ \text{Li}_2(z) = \text{Li}_1(z) \otimes \log z$, then

$$\text{Disc}(\text{Li}_2(z)) = \mu\ (\text{Disc} \otimes \text{id})\ (\text{Li}_1(z) \otimes \log z) = 2\pi i\ \log z\,, \tag{2.160}$$

From the coproduct on classical polylogarithms,

$$\Delta_{1,n-1}(\text{Li}_n(z)) = \text{Li}_1(z) \otimes \frac{\log^{n-1} z}{(n-1)!}\,, \tag{2.161}$$

then

$$\text{Disc}(\text{Li}_n(z)) = \mu\ (\text{Disc} \otimes \text{id})\ \left(\text{Li}_1(z) \otimes \frac{\log^{n-1} z}{(n-1)!}\right) = 2\pi i\ \frac{\log^{n-1} z}{(n-1)!}\,. \tag{2.162}$$

2.6.9 The symbol map

The symbol may be defined as a total differential. In fact, introducing a transcendental function F_n of weight n as a $\mathbb{Q}$-linear combination of n-fold iterated integrals,

$$F_n = \int_a^b \mathrm{dlog} R_1 \circ \cdots \circ \mathrm{dlog} R_n = \int_a^b \Big(\int_a^t \mathrm{dlog} R_1 \circ \cdots \circ \mathrm{dlog} R_{n-1} \Big) \mathrm{dlog} R_n(t) \,, \tag{2.163}$$

where R_i are rational functions, the total differential of F_n is

$$dF_n = \sum_i F_{i,n-1} \, \mathrm{dlog} R_i \,, \tag{2.164}$$

where $F_{i,n-1}$ are transcendental functions of weight $n-1$. Then the symbol can be computed recursively,

$$S(F_n) = \sum_i S(F_{i,n-1}) \otimes \log R_i \,. \tag{2.165}$$

For generic values of a_i, the total differential of a MPL is [47]

$$dI(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{i=1}^{n} I(a_0; a_1, \ldots, \not{a}_i, \ldots, a_n; a_{n+1}) \; \mathrm{dlog}\Big(\frac{a_{i+1} - a_i}{a_{i-1} - a_i}\Big) \,, \tag{2.166}$$

where $\not{a}_i$ indicates the missing element. The symbol of the MPL is then

$$S(I(a_0; a_1, \ldots, a_n; a_{n+1})) = \sum_{i=1}^{n} S(I(a; a_1, \ldots, \not{a}_i, \ldots, a_n; a_{n+1})) \otimes \log\Big(\frac{a_{i+1} - a_i}{a_{i-1} - a_i}\Big) \,. \tag{2.167}$$

Since the symbol is of the type $\log(a_1) \otimes \ldots \otimes \log(a_n)$ it is customary to use the short-hand $a_1 \otimes \ldots \otimes a_n$. The entries $a_1, \ldots, a_n$ of the symbol are called the letters and the set of entries $\{a_1, \ldots, a_n\}$ the alphabet of the function F_n.

The symbol has the properties of the logarithm,

$$\begin{aligned} \cdots \otimes (ab) \otimes \cdots &= \cdots \otimes a \otimes \cdots + \cdots \otimes b \otimes \cdots \\ \cdots \otimes a^n \otimes \cdots &= n \, (\cdots \otimes a \otimes \cdots) \,, \end{aligned} \tag{2.168}$$

which implies that $\cdots \otimes 1 \otimes \cdots = 0$. Further, if we have a transcendental function, $F_n(z_1, \ldots, z_k)$ of weight n, whose symbol is

$$S(F_n(z_1, \ldots, z_k)) = \sum_{i_1, \ldots, i_n} c_{i_1 \ldots i_n} \; a_{i_1} \otimes \cdots \otimes a_{i_n} \,, \tag{2.169}$$

with $c_{i_n \ldots i_n} \in \mathbb{Q}$, and a_i are rational functions of $(z_1, \ldots, z_n)$, then the derivative acts on the last entry of the symbol,

$$S\Big(\frac{\partial}{\partial z_j} \, F_n(z_1, \ldots, z_k)\Big) = \sum_{i_1, \ldots, i_n} c_{i_1 \ldots i_n} \; a_{i_1} \otimes \cdots \otimes a_{i_{n-1}} \, \frac{\partial}{\partial z_j} a_{i_n} \,. \tag{2.170}$$

If $\text{Disc}|_{z_j=p}$ is the discontinuity of F around the branch point $z_j = p$, then

$$S\big(\text{Disc } F(z_1,\ldots,z_k)|_{z_j=p}\big) = \sum_{i_1,\ldots,i_n} \text{Disc } \log a_{i_1}|_{z_j=p} \cdot c_{i_1\ldots i_n}\, a_{i_2} \otimes \cdots \otimes a_{i_n}\,, \tag{2.171}$$

where

$$\text{Disc } \log a_{i_1}|_{z_j=p} = \begin{cases} 2\pi i, & \text{if } a_{i_1} \text{ has a zero for } z_j = p, \\ 0, & \text{otherwise.} \end{cases} \tag{2.172}$$

However the symbol is blind to $i\pi$, so if $\rho^n - 1 = 0$ and ρ_n is the n-th root of unity,

$$\cdots \otimes \rho_n \otimes \cdots = 0\,. \tag{2.173}$$

Likewise, it is blind to MZVs,

$$\cdots \otimes \zeta_{m_1,\ldots,m_k} \otimes \cdots = 0\,, \tag{2.174}$$

i.e. the symbol map is not injective. In fact, besides factors of $i\pi$ and MZVs, the kernel of the symbol map contains also combinations of transcendental numbers [54].

We note that, up to $i\pi$ terms, the symbol is equivalent to the maximal iteration of the coproduct of a function F_n of weight n,

$$S(F_n) = \Delta_{\underbrace{1,\ldots,1}_{n}} F_n \quad \text{mod } i\pi\,, \tag{2.175}$$

which is an n-fold tensor product of functions of weight one, i.e. logarithms, which can also be written recursively, like the symbol,

$$\Delta_{1,\ldots,1}(F_n) = \sum_i \Delta_{1,\ldots,1}(F_{i,n-1}) \otimes \log R_i\,, \tag{2.176}$$

and whose differentiation is

$$(\text{id} \otimes d)\; \Delta_{1,\ldots,1}(F_n) = \sum_i \Delta_{1,\ldots,1}(F_{i,n-1}) \otimes \text{dlog} R_i\,, \tag{2.177}$$

which is consistent with the definition (2.167) of the symbol as a total differential.

Finally, since the symbol is included in a coproduct element, it inherits all the properties of the coproduct, like e.g. the shuffle product,

$$S(F\, G) = S(F) \;\sqcup\!\sqcup\; S(G)\,. \tag{2.178}$$

The symbol is of great interest, because being related to the total differential of an iterated integral, it is linked to the canonical form of the differential equations. Further, the total differential of F_n has one less weight than F_n, so it is often easier to obtain than F_n itself. For example, for many amplitudes in planar $N = 4$ SYM, the symbol of the amplitude is known, but the amplitude itself is not.

Finally, a necessary condition for two expressions written in terms of MPLs to be equal is that they have the same symbol, but it is not sufficient because the symbol, being blind to $i\pi$ terms and to MZVs, has a loss of information, in fact the maximal loss among coproduct elements.

Given a tensor,

$$T = \sum_{i_1,\ldots,i_k} c_{i_1\ldots i_k}\, a_{i_1} \otimes \cdots \otimes a_{i_k}, \quad \text{with } c_{i_1\ldots i_k} \in \mathbb{Q}\,, \tag{2.179}$$

a function F, such that $S(F) = T$, exists if and only if T satisfies the integrability condition,

$$\sum_{i_1,\ldots,i_k} c_{i_1\ldots i_k}\, \text{dlog}\, a_{i_j} \wedge \text{dlog}\, a_{i_{j+1}}\, a_{i_1} \otimes \cdots \otimes a_{i_{j-1}} \otimes a_{i_{j+2}} \cdots \otimes a_{i_k} = 0\,, \tag{2.180}$$

for all $1 \leq j \leq k-1$, and where $\wedge$ is the wedge product on differential forms. However, there is no algorithm to construct such a function F.

2.6.10 The functional equations for multiple polylogarithms

We discuss now how the Hopf algebra helps in deriving functional equations for MPLs, which are the third-millennium analog of the logarithmic tables. In order to examine the relation between two functions F_n and G_n of weight n, we decompose them into lower-weight functions using the coproduct, and exploit the relations among the latter, assuming that they are known.

We assume that if the two functions F_n and G_n share the same reduced coproduct,

$$\Delta^{'}(F_n) = \Delta^{'}(G_n)\,, \tag{2.181}$$

then

$$F_n = G_n + \sum_i c_i\, P_{i,n}\,, \tag{2.182}$$

where $c_i \in \mathbb{Q}$ and the $P_{i,n}$ are constant primitive elements of weight n. The $P_{i,n}$ may be powers of π, ζ_n values and Clausen values of the roots of unity,

$$Cl_n\Big(\frac{k\pi}{N}\Big) = \begin{cases} \text{Re}\big(\text{Li}_n(e^{ik\pi/N})\big), & \text{even } n \\ \text{Im}\big(\text{Li}_n(e^{ik\pi/N})\big), & \text{odd } n \end{cases} \tag{2.183}$$

The undetermined constants can then be fixed by evaluating F_n and G_n at fixed points.

We display the procedure on the inversion relation. At weight 1, we know that

$$\text{Li}_1\Big(\frac{1}{z}\Big) = -\text{log}\Big(1-\frac{1}{z}\Big) = -\text{log}(1-z) + \text{log}(-z) = -\text{log}(1-z) + \log z - i\pi\,. \tag{2.184}$$

At weight 2, we take the coproduct element,

$$
\begin{aligned}
\Delta_{1,1}\big(\mathrm{Li}_2\big(\frac{1}{z}\big)\big) &= \mathrm{Li}_1\big(\frac{1}{z}\big)\otimes\log\big(\frac{1}{z}\big) \\
&= \log(1-z)\otimes\log z - \log z\otimes\log z + i\pi\otimes\log z \\
&= \Delta_{1,1}\big(-\mathrm{Li}_2(z) - \frac{1}{2}\,\log^2 z + i\pi\log z\big)\,, \qquad (2.185)
\end{aligned}
$$

where we have used the inversion relation (2.184), and that $\Delta(i\pi) = i\pi\otimes\mathbb{1}$. Note that with the symbol, we would have missed the $i\pi$ term. Since $\Delta^{'}(F_2) = \Delta^{'}(G_2)$, the arguments on the left- and right- hand sides must be equal up to primitive elements, which are supposed to be weight-two constants, with coefficients to be determined,

$$
\mathrm{Li}_2\big(\frac{1}{z}\big) = -\mathrm{Li}_2(z) - \frac{1}{2}\,\log^2 z + i\pi\log z + a\zeta_2\,, \qquad (2.186)
$$

with $a\in\mathbb{Q}$. Setting $z=1$, we have

$$
\zeta_2 = -\zeta_2 + a\zeta_2 \Rightarrow a = 2, \qquad (2.187)
$$

thus

$$
\mathrm{Li}_2\big(\frac{1}{z}\big) = -\mathrm{Li}_2(z) - \frac{1}{2}\,\log^2 z + i\pi\log z + 2\zeta_2\,. \qquad (2.188)
$$

At weight three, we take the coproduct element,

$$
\begin{aligned}
&\Delta_{1,1,1}\big(\mathrm{Li}_3\big(\frac{1}{z}\big)\big) = \mathrm{Li}_1\big(\frac{1}{z}\big)\otimes\log\big(\frac{1}{z}\big)\otimes\log\big(\frac{1}{z}\big) \\
&= -\log(1-z)\otimes\log z\otimes\log z + \log z\otimes\log z\otimes\log z - i\pi\otimes\log z\otimes\log z \\
&= \Delta_{1,1,1}\big(\mathrm{Li}_3(z) + \frac{1}{6}\,\log^3 z - \frac{i\pi}{2}\,\log^2 z\big)\,. \qquad (2.189)
\end{aligned}
$$

The missing terms, that cannot be detected by $\Delta_{1,1,1}$ must be of type $\zeta_2\log z$, ζ_3, $(i\pi)^3$. So, in order to catch $(\zeta_2\log z)$ terms, we look at $\Delta_{2,1}$,

$$
\begin{aligned}
&\Delta_{2,1}\big(\mathrm{Li}_3\big(\frac{1}{z}\big) - \big(\mathrm{Li}_3(z) + \frac{1}{6}\,\log^3 z - \frac{i\pi}{2}\,\log^2 z\big)\big) \\
&= \mathrm{Li}_2\big(\frac{1}{z}\big)\otimes\log\big(\frac{1}{z}\big) - \mathrm{Li}_2(z)\otimes\log z - \frac{1}{2}\,\log^2 z\otimes\log z + (i\pi\log z)\otimes\log z \\
&= -\left(-\cancel{\mathrm{Li}_2(z)} - \cancel{\frac{1}{2}\,\log^2 z} + \cancel{i\pi\log z} + 2\zeta_2\right)\otimes\log z \\
&\quad -\cancel{\mathrm{Li}_2(z)}\otimes\log z - \frac{1}{2}\,\cancel{\log^2 z}\otimes\log z + \cancel{(i\pi\log z)}\otimes\log z \\
&= -2\,\zeta_2\otimes\log z \\
&= \Delta_{2,1}(-2\,\zeta_2\,\log z)\,, \qquad (2.190)
\end{aligned}
$$

where we have used the inversion relation (2.188), so

$$\mathrm{Li}_3\Big(\frac{1}{z}\Big) = \mathrm{Li}_3(z) + \frac{1}{6}\,\log^3 z - \frac{i\pi}{2}\,\log^2 z - 2\zeta_2 \log z + a\zeta_3 + bi\pi^3\,, \tag{2.191}$$

with $a, b \in \mathbb{Q}$. Setting $z = 1$, we have

$$\zeta_3 = \zeta_3 + a\zeta_3 + bi\pi^3 \Rightarrow a = b = 0, \tag{2.192}$$

so

$$\mathrm{Li}_3\Big(\frac{1}{z}\Big) = \mathrm{Li}_3(z) + \frac{1}{6}\,\log^3 z - \frac{i\pi}{2}\,\log^2 z - 2\zeta_2 \log z\,. \tag{2.193}$$

2.6.11 Single-valued multiple polylogarithms

Single-valued functions are real analytic functions on the complex plane. Conversely, classical polylogarithms are multi-valued functions, of which in eq. (2.162) we computed the discontinuity,

$$\mathrm{Disc}(\mathrm{Li}_n(z)) = 2\pi i\,\frac{\log^{n-1} z}{(n-1)!}\,.$$

However, one can build linear combinations of classical polylogarithms, such that all branch cuts cancel. The single-valued classical polylogarithm [55] is

$$P_n^{SV} = R_n\,\Big[\sum_{k=0}^{n-1} \frac{2^k B_k}{k!}\,\log^k|z|\;\mathrm{Li}_{n-k}(z)\Big]\,, \qquad R_n = \begin{cases} \mathrm{Re}, & \text{odd } n \\ \mathrm{Im}, & \text{even } n \end{cases} \tag{2.194}$$

and with B_k the Bernoulli numbers.

Likewise, one can construct single-valued versions of the HPLs, and more in general, of the MPLs. E.g. the scalar integral of the three-mass triangle that we mentioned in eq. (2.61) can be written in terms of single-valued MPLs [56], which are functions of z, $\bar{z}$, defined as in eq. (2.68).

Single-valued MPLs form a shuffle algebra and a Hopf algebra, like the MPLs. In the last decade, single-valued MPLs have been used to describe many amplitudes in QCD and in $N = 4$ SYM, like e.g. the QCD soft anomalous dimension, amplitudes in multi-Regge kinematics, and the energy flow of QCD jets.

2.7 Diagrammatic coaction

The coefficients of the Laurent expansion of a Feynman integral are periods. In the one-loop case, these periods are MPLs, and it is possible to construct a (diagrammatic) coaction acting on one-loop integrals, which works like the coaction on MPLs, order by order in the Laurent expansion [57].

The coaction on a one-loop integral can be written in such a way that all the first entries are

Feynman integrals, while the second entries are cuts of the original integral, e.g. on a massless bubble,

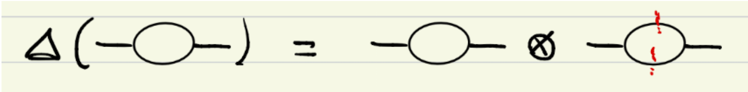
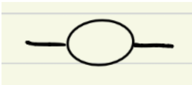


Figure 2.16: Coaction on the massless bubble.

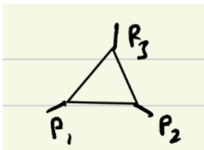
with  $= -\frac{c_\Gamma}{\epsilon}\,(-p^2)^{-\epsilon}\,,$

where c_Γ is an ubiquitous one-loop factor,

$$c_\Gamma = \frac{e^{\gamma_E \epsilon}\,\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\,, \tag{2.195}$$

and  $= -\frac{e^{\gamma_E \epsilon}\,\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\,(p^2)^{-\epsilon}\,.$

Further, on the three-mass triangle,



that we introduced in eq. (2.61), the reduced coaction (i.e. without the two trivial terms) is

$$\begin{aligned}&\Delta^{'}(\text{3-mass triangle})\\ &= \log(-p_1^2)\otimes\log(\frac{\bar{z}(1-z)}{z(1-\bar{z})}) + \log(-p_2^2)\otimes\log\frac{1-\bar{z}}{1-z} + \log(-p_3^2)\otimes\log\frac{z}{\bar{z}}\,,\end{aligned} \tag{2.196}$$

which shows that in the case of massless propagators all the first entries of weight one in the coaction are logarithms of Mandelstam invariants. This is called first-entry condition. First entries are related to discontinuities and reflect the fact, that we already treated when discussing unitarity in sec. 2.1, that Feynman integrals with massless propagators can only have branch points when Mandelstam invariants vanish or are infinite.

Since branch points or singularities occur when the Landau equations are satisfied, the diagrammatic coaction formulation allows one to characterise thoroughly the pinch-singularity surfaces at one loop.

2.8 Differential equations

The integration-by-part identities (see app. E and app. H.49) allow one to reduce the scalar integrals to a linearly independent set of scalar integrals called master integrals. The differential equations in the external parameters (masses or Mandelstam invariants) provide a method of computing the master integrals.

For n external parameters x_j and k master integrals f_B, one obtains a system of n first-order partial differential equations (PDE),

$$\partial_i f_B(x_j;\epsilon) = A^{(i)}_{BC}(x_j;\epsilon)\; f_C(x_j;\epsilon)\,, \tag{2.197}$$

with $i = 1,\dots,n$ and $B,C = 1,\dots,k$ and where A_i are $k \times k$ matrices, whose entries are rational functions of the external parameters and of ϵ.

On the example of Higgs production from gluon fusion, which is mediated by a top-quark loop, in app. H.50 you have seen all the main features of the system of PDEs, namely that:

- It fulfills an integrability condition,

$$\partial_i A^{(j)} - \partial_j A^{(i)} + [A^{(j)}, A^{(i)}] = 0\,. \tag{2.198}$$

  Note that the system of PDEs can also be cast as a differential form, $df + A(x;\epsilon)f = 0$, where $A = \sum_{i=1}^{n} A^{(i)}\, dx_i$ is a matrix-valued one-form. Then the integrability condition is $dA + A \wedge A = 0$.

- it is possible to rotate the basis of master integrals,

$$\vec{f'} = T(x;\epsilon)\; \vec{f}\,, \tag{2.199}$$

  where $T(x;\epsilon)$ is a $k \times k$ matrix. Then

$$\tilde{A}^{(i)} = (\partial_i T)\; T^{-1} + T\; A^{(i)}\; T^{-1}\,, \tag{2.200}$$

  In differential forms, $\tilde{A}$ is a connection,

$$\tilde{A} = (dT)\; T^{-1} + T\; A\; T^{-1} \tag{2.201}$$

If $A^{(i)}$ can be rotated to an $\epsilon -$ independent form,

$$\tilde{A}^{(i)}(x;\epsilon) = \epsilon \bar{A}^{(i)}(x_j)\,, \tag{2.202}$$

with

$$\bar{A}^{(i)}(x_j) = \sum_{\ell=1}^{L} C^{(i)}_{\ell} w_\ell\,, \tag{2.203}$$

where $C^{(i)}_{\ell}$ are $k \times k$ matrices $(C^{(i)}_{\ell})_{AB}$, whose entries are rational numbers, w_ℓ are L $\mathbb{Q}$-linear independent differential forms, called the letters, with only simple poles in x, the system of PDEs becomes

$$\partial_i \vec{g}(x_j;\epsilon) = \epsilon\; \partial_i \bar{A}^{(i)}(x_j)\; \vec{g}(x_j;\epsilon)\,, \tag{2.204}$$

and it is said to be in canonical form [58]. In app. H.51, the system of PDEs in canonical form is worked out on the example of Higgs production from gluon fusion.

Parametrising the differential forms, $\gamma^*(w_i) = p_i(\lambda)\, d\lambda$, on the unit interval $[0, 1]$, such that a line integral is given by,

$$\int_\gamma w_i = \int_{(0,1)} \gamma^*(w_i) = \int_0^1 d\lambda\ p_i(\lambda)\,, \tag{2.205}$$

the solution $\vec{g}(x_j;\epsilon)$ can be written as

$$\vec{g}(x_j;\epsilon) = P\ \exp\left(\epsilon \int_\gamma d\bar{A}\right)\ \vec{g}_0(\epsilon)\,, \tag{2.206}$$

where P is the path ordering on the integration contour γ, and $\vec{g}_0$ is a boundary term. Eq. (2.206) is to be expanded in ϵ ,where the n-th term in the expansion is a $\mathbb{Q}$-linear combination of n-fold iterated integrals,

$$\int_\gamma w_1 \cdots w_n = \int_{0\le\lambda_n\le\ldots\le\lambda_1\le 1} d\lambda_1\ p_1(\lambda_1)\cdots d\lambda_n\ p_n(\lambda_n)\,. \tag{2.207}$$

If $w_i = \mathrm{dlog}(a_i(x))$, where a_i are polynomials in x, then $p_i(\lambda)\ d\lambda = \dfrac{d\lambda}{\lambda - z_i}$, where z_i are the singularities, and the iterated integrals are MPLs,

$$\int_\gamma w_1 \cdots w_n = \int_0^\lambda \frac{d\lambda_1}{\lambda_1 - z_1} \int_0^{\lambda_1} \frac{d\lambda_2}{\lambda_2 - z_2} \cdots \int_0^{\lambda_{n-1}} \frac{d\lambda_n}{\lambda_n - z_n} = G(z_1, \ldots, z_n; \lambda)\,. \tag{2.208}$$

In practice, when the system is in canonical form (2.204), we Taylor expand the vector of master integrals,

$$\vec{g}(x_j;\epsilon) = \sum_{m=0}^{\infty} \epsilon^m \vec{g}_m(x_j)\,, \tag{2.209}$$

and the system of PDEs is solved recursively in ϵ,

$$\partial_i \vec{g}_m(x_j) = \partial_i \bar{A}^{(i)}(x_j)\ \vec{g}_{m-1}(x_j)\,, \qquad m \ge 1\,. \tag{2.210}$$

Note also that in some cases, usually when there are massive propagators, like in Higgs production from gluon fusion examined in app. H.51, the rotation to an ϵ-independent form introduces algebraic factors (square roots) in the A matrices. Then a (coordinate) transformation of the external parameters base $(x_1, \ldots, x_n)$,

$$A = \sum_{i=1}^{n} A^{(i)} dx_i \to A' = \sum_{i,j=1}^{n} A^{(i)} \frac{\partial x_i}{\partial x'_j} dx'_j\,, \tag{2.211}$$

is necessary to put back the A matrices in rational form.

In fact, in Higgs production from gluon fusion, after the rotation to an ϵ-independent form and a change of variables $\dfrac{m_H^2}{m_t^2} = -\dfrac{(1-x)^2}{x}$ to rationalise back the A matrices, there is one external parameter, x, three master integrals, and two letters, x and $1 + x$, and the differential equation in canonical form is

$$\partial_x \vec{g}(x;\epsilon) = \epsilon\ \partial_x \bar{A}(x)\ \vec{g}(x;\epsilon)\,, \tag{2.212}$$

with

$$\bar{A}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \log x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \log(1+x)\,, \tag{2.213}$$

$$\partial_x \bar{A}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \frac{1}{x} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{1+x}\,. \tag{2.214}$$

Since the letters are x and $1+x$, $\vec{g}(x;\epsilon)$ can be written in terms of HPLs (2.81), see app. H.51.

In app. H.51, you have seen how working on the maximal cut, where all sub-topologies vanish, helps in finding the canonical form. The big questions are: when can we put a system of PDEs in canonical form? And how?

There are no (as yet) general answers.

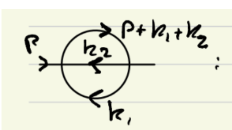
In the case of Feynman integrals evaluating to MPLs, there are algorithms to find a rotation $\vec{f'} = T(x;\epsilon)\ \vec{f}$ of the basis of master integrals to achieve the canonical form, provided that $T(x;\epsilon)$ is rational in the external parameters x. There are also algorithms to transform the external parameter base $(x_1, \ldots, x_n)$, in order to rationalise the square roots. But those algorithms have limitations.

To further complicate the scenario, not all Feynman integrals evaluate to MPLs. There are also Feynman integrals associated to elliptic curves. In that case, an ϵ-independent form of $A^{(i)}$ can sometimes be achieved.

2.9 Elliptic integrals and elliptic curves

We explore here a topic which is at the forefront of research in Amplitudes: elliptic integrals. As discussed in sec. 2.5, the coefficients of the Laurent expansion of a Feynman integral are periods. In the one-loop case, those periods are MPLs. At two loops and beyond, those periods may also be elliptic integrals. This usually happens when there are internal masses, i.e. massive propagators, in the loops. But it also occurs in massless two-loop scalar integrals with ten or more legs. Elliptic integrals were introduced first in 1962 by Sabry in the two-loop electron self-energy. They reached maturity in the last decade, but it is only in the last three years that a systematic study of their properties has been undertaken.

Let us look at the paradigm case: the equal-mass sunrise integral, with $p^2 = s$,



$$= \int d^d k_1\ d^d k_2\ \frac{1}{(k_1^2 - m^2)(k_2^2 - m^2)((p+k_1+k_2)^2 - m^2)}\,.$$

We introduce the Feynman parametrisation (2.53), perform the loop integrals and, up to an overall factor, we get

$$I = \int_0^1 da_1 \; da_2 \; da_3 \; \delta(1 - a_1 - a_2 - a_3) \; U^{3-\frac{3d}{2}} \; F^{d-3} \,, \tag{2.215}$$

$$\text{where} \qquad \begin{cases} U = a_1 a_2 + a_2 a_3 + a_3 a_1 \\ F = (a_1 + a_2 + a_3) \; U \; m^2 - a_1 a_2 a_3 \; s \,, \end{cases} \tag{2.216}$$

Note that U and F are homogeneous polynomials of degree two and three respectively, so the integration on the domain $0 \leq a_i \leq 1$ with $a_1 + a_2 + a_3 = 1$, can also be performed on the projective domain,

$$\sigma = \{[a_1 : a_2 : a_3] \in \mathbb{P}^3 \; / \; a_i \geq 0\} \,. \tag{2.217}$$

A theorem due to Cheng and Wu states that a projective Feynman integral has the same value when integrated over the domain,

$$\sigma = \{a_i \geq 0 \; / \; \sum_{i \in S} a_i = 1\} \,, \tag{2.218}$$

where S is a non-empty subset of $\{a_1, \ldots, a_n\}$. In practice, we may choose $S = \{a_1\}$, do the integral $\int da_1 \; \delta(1 - a_1)$ by setting $a_1 = 1$, and we get

$$I = \int_0^\infty da_2 \; da_3 \; U^{3-\frac{3d}{2}} \; F^{d-3} \,, \tag{2.219}$$

with

$$\begin{aligned} U &= a_2 + a_3 + a_2 a_3 \,, \\ F &= (1 + a_2 + a_3) \; U \; m^2 - a_2 a_3 \; s \,. \end{aligned} \tag{2.220}$$

Dimensional-shift relations relate integrals in d dimensions to integrals in $(d-2)$ dimensions. In the sunrise integral, it is convenient to compute the integral in $2 - 2\epsilon$ dimensions because its Laurent expansion has no ϵ poles in $2 - 2\epsilon$, $I = \sum_{n=0}^{\infty} I_n \; \epsilon^n$, and because the I_0 term in the expansion has no U term,

$$I = \int_0^\infty da_2 \int_0^\infty da_3 \; \frac{1}{F} \,, \tag{2.221}$$

We can view F as a second-order polynomial in a_2,

$$F = A \; a_2^2 + B \; a_2 + C = A \; (a_2 - a_+)(a_2 - a_-) \,, \tag{2.222}$$

with

$$
\begin{aligned}
a_{\pm} &= \frac{a_3 s - (a_3^2 + 3\ a_3 + 1)\ m^2 \pm \sqrt{D(a_3)}}{2(a_3+1)\ m^2}\,, \\
D(a_3) &= (a_3^2 + a_3 + 1)^2\ m^4 - 2a_3(a_3^2 + 3\ a_3 + 1)\ m^2 s + a_3^2\ s^2\,,
\end{aligned} \tag{2.223}
$$

so we get

$$
\begin{aligned}
\frac{1}{F} &= \frac{1}{A\ (a_2 - a_+)(a_2 - a_-)} = \frac{1}{A}\ \frac{1}{a_+ - a_-}\Big(\frac{1}{a_2 - a_+} - \frac{1}{a_2 - a_-}\Big) \\
&= \frac{1}{\sqrt{D(a_3)}}\ \Big(\frac{1}{a_2 - a_+} - \frac{1}{a_2 - a_-}\Big)\,.
\end{aligned} \tag{2.224}
$$

Then we can do easily the integrals over a_2. The contributions at $a_2 \to \infty$ cancel each other, and we get

$$
I_0 = \int_0^\infty da_3\ \frac{\log\Big(\frac{a_3\ s - (a_3^2 + 3\ a_3 + 1)\ m^2 - \sqrt{D(a_3)}}{a_3\ s - (a_3^2 + 3\ a_3 + 1)\ m^2 + \sqrt{D(a_3)}}\Big)}{\sqrt{D(a_3)}}\,, \tag{2.225}
$$

where $D(a_3)$ is a degree-four polynomial in a_3. This is an elliptic integral.

The name comes from the integral over an arc length of an ellipse. In sec. 2.5, we said that the perimeter of an ellipse with radii a and b is

$$
I = 4 \int_0^b d\eta\ \sqrt{1 + \frac{a^2\ \eta^2}{b^4 - b^2\ \eta^2}}\,, \qquad \text{with } b > a\,. \tag{2.226}
$$

Changing variable $\eta = b\ x$, we can also write it as

$$
\begin{aligned}
I &= 4b\ \int_0^1 dx\ \sqrt{\frac{1 - k^2 x^2}{1 - x^2}}, \qquad \text{with } k^2 = 1 - \frac{a^2}{b^2} \\
&= 4b\ \int_0^1 dx\ \frac{1 - k^2 x^2}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}\,.
\end{aligned} \tag{2.227}
$$

For $k^2 = 0$, we get the perimeter of the circumference, $4\ \int_0^1 dy\ (1 - x^2)^{-1/2}$,

So the problem of determining an arc length of an ellipse is equivalent to evaluating the integral,

$$
\int dx\ \frac{1 - k^2 x^2}{y(x)}\,, \qquad \text{with } y^2(x) = (1 - x^2)(1 - k^2 x^2)\,, \tag{2.228}
$$

which is called elliptic integral. Since the square root of a quartic polynomial appears in the denominator, by analogy, the square root of a quartic polynomial, with different roots is called elliptic curve, although, per se, it has nothing to do with an ellipse. Cubic and quartic polynomials with different roots yield elliptic curves, which we list here in the most used forms,

$$y^2 = x^3 + A\,x + B\,, \qquad \text{Weierstrass form}\,, \tag{2.229}$$

$$y^2 = x(x-1)(x-\lambda)\,, \qquad \text{Legendre form}\,, \tag{2.230}$$

$$y^2 = (1-x^2)(1-k^2\,x^2)\,, \qquad \text{Jacobi form}\,. \tag{2.231}$$

In the app. G we review the elliptic curve in Weierstrass form. In app. H.61, we see how to pass from the Weierstrass form to the Legendre and Jacobi forms.

Let us consider an elliptic curve,

$$E:\; y^2 = x(x-1)(x-\lambda)\,, \tag{2.232}$$

in Legendre form over $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \approx \mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C})$, which is topologically a 2-sphere, and let us take the holomorphic differential form, $\omega = \frac{dx}{y}$. The integral,

$$\int_O^P \omega = \int_\infty^x \frac{dt}{\sqrt{t(t-1)(t-\lambda)}}\,, \tag{2.233}$$

is path dependent, because the square root is double valued. The branch points are at $0, 1, \lambda, \infty$. We may glue together two copies of $\mathbb{CP}^1$ along the branch cuts and form a torus, which has genus 1, as seen below in fig. (2.17).

Figure 2.17: Left panel: Sphere with the location of the branch points on it. Right panel: Two copies of the sphere glued together to form a torus.

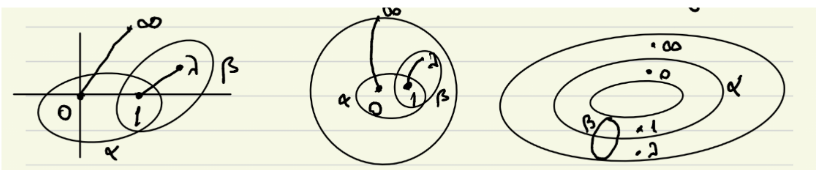


Figure 2.18: Closed contours on the complex plane, on the sphere and on the torus.

The integrals on the closed contours α and β,

$$\omega_1 = \oint_\alpha \omega, \qquad \omega_2 = \oint_\beta \omega\,, \tag{2.234}$$

are (non-zero) complex numbers. They are called the periods of the elliptic curve (not to be confused with the periods as numbers).

The k-th homology group $H_k(X)$ of a topological space X describes basically the k-dimensional holes in X. The circle S_1, has a one-dimensional hole.

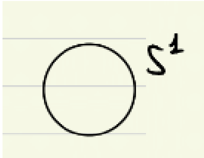


Figure 2.19: Circle S_1.

The first homology group of the circle is $H_1(S_1) = \mathbb{Z}$, i.e. an abelian group, which is generated by a closed contour α around the one-dimensional hole. Then the elements of $H_1(S_1)$ can be written as $n\alpha$, with $n \in \mathbb{Z}$, i.e. any two paths on S_1 differ by a path which is homologous to $n\alpha$, and any integral $\int_O^p \omega$ is defined up to the addition of $n \oint_\alpha \omega$.

Since the torus T may be defined as the product of two circles, $T = S^1 \times S^1$, it has two independent one-dimensional holes and its first homology group is $H_1(T) = \mathbb{Z} \times \mathbb{Z}$, which is generated by the closed contours a and β of eq. (2.234). Thus any two paths on T differ by a path which is homologous to $n_1\alpha + n_2\beta$, for $n_1, n_2 \in \mathbb{Z}$, and any integral,

$$\int_O^P \omega = \int_\infty^x \frac{dt}{\sqrt{t(t-1)(t-\lambda)}}, \tag{2.235}$$

is defined up to the addition of $n_1\omega_1 + n_2\omega_2$.

When the periods ω_1 and ω_2 are $\mathbb{R}$-linearly independent, i.e. there is no $\lambda \in \mathbb{R}$, such that $\omega_1 = \lambda\omega_2$, we define the lattice,

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 \ / \ n_1, n_2 \in \mathbb{Z}\}. \tag{2.236}$$

Λ is a subgroup of $\mathbb{C}$, and the quotient space $\mathbb{C}/\Lambda$, i.e. the torus, is a group. The fundamental cell D is the parallelogram spanned by the periods of the lattice,

$$D = \{z + t_1\omega_1 + t_2\omega_2 \ / \ 0 \leq t_1, \ t_2 \leq 1\}. \tag{2.237}$$

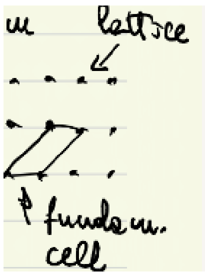


Figure 2.20: A graphical representation of a lattice with its fundamental cell.

Meromorphic functions on $\mathbb{C}/\Lambda$ are meromorphic functions on $\mathbb{C}$, which are (doubly) periodic with respect to Λ. An elliptic function, with respect to Λ, is a meromorphic function $f(z)$ on $\mathbb{C}$ which is invariant under translations by the periods ω_i,

$$f(z+\omega) = f(z), \qquad \forall\, z \in \mathbb{C},\ \forall \omega \in \Lambda\,. \tag{2.238}$$

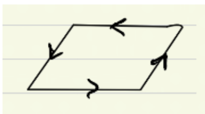
The set of elliptic functions is $\mathbb{C}(\Lambda)$, and it is a field. A holomorphic elliptic function, i.e. one with no poles, is constant. In fact, since the function is holomorphic, it is bounded on the fundamental cell, which is a compact set, but then due to its periodicity it is bounded on all of $\mathbb{C}$, and by Liouville's theorem it is constant. Similarly, an elliptic function with no zeros is constant.

Given an elliptic function $f(w)$, with $w \in \mathbb{C}$, we can examine its residue and its order of vanishing, i.e. its number of zeros. Since f is elliptic, the residue and the order of vanishing do not change if we replace w, by $w+\omega$, $\forall\, \omega \in \Lambda$. So it is enough to consider the residue and the order of vanishing on the fundamental cell D.

We choose the fundamental cell D of Λ, such that $f(z)$ has no zeros or poles on the boundary ∂D of D. The residue theorem implies that

$$\sum_{w \in \mathbb{C}/\Lambda} \mathrm{Res}_w(f) = \frac{1}{2\pi i} \int_{\partial D} dz\; f(z)\,. \tag{2.239}$$

Since f is periodic in ω_1 and in ω_2, $f(z+\omega) = f(z)$, the integrals along the opposite sides of the cell cancel, so the integral around the boundary ∂D vanishes,



$$\sum_{w \in \mathbb{C}/\Lambda} \mathrm{Res}_w(f) = 0\,. \tag{2.240}$$

Also $f'(z)$ is periodic, and since $\mathrm{Res}_w\left(\dfrac{f'}{f}\right) = \mathrm{ord}_w(f)$, we have that

$$\sum_{w \in \mathbb{C}/\Lambda} \mathrm{Res}_w\left(\frac{f'}{f}\right) = \sum_{w \in \mathbb{C}/\Lambda} \mathrm{ord}_w(f) = 0\,, \tag{2.241}$$

i.e. the number of zeros equals the number of poles.

The order of an elliptic function is its number of poles (or zeros), weighted by multiplicity, in the cell D. A non-constant elliptic function must have at least two poles or a double pole, because if it had a single simple pole, eq. (2.240) implies that its residue vanishes, thus the function f is holomorphic, and hence constant. So a non-constant elliptic function has order ≥ 2.

We introduce the Weierstrass $\wp$ function on Λ,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda,\ \omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \tag{2.242}$$

and the Eisenstein series of weight $2k$ on Λ,

$$G_{2k} = \sum_{\omega\in\Lambda,\ \omega\neq 0} \omega^{-2k} \,. \tag{2.243}$$

Theorem 1: (the proof is given in Silverman's book [59])

- G_{2k} is absolutely convergent $\forall\ k > 1$.
- $\wp(z)$ is absolutely and uniformly convergent on every compact subset of $\mathbb{C}\backslash\Lambda$. The series defines a meromorphic function on $\mathbb{C}$ with a double pole with null residue at each lattice point, and no other poles.
- $\wp(z)$ is an even elliptic function.

The derivative of the Weierstrass $\wp$ function is

$$\wp'(z) = -2 \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3} \,, \tag{2.244}$$

$\wp'(z)$ is also an elliptic function, $\wp'(z+\omega) = \wp'(z)$.

Theorem 2: (proof in Silverman's book [59]).

For a lattice $\Lambda \subset \mathbb{C}$,

$$\mathbb{C}(\Lambda) = \mathbb{C}(\wp(z), \wp'(z)) \,, \tag{2.245}$$

i.e. every elliptic function is a rational combination of $\wp(z)$ and $\wp'(z)$.

Theorem 3: (see app. H.62)

The Laurent series for $\wp(z)$ around $z = 0$ is

- $\wp(z) = \dfrac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\ G_{2k+2}\ z^{2k}$.
- $\forall\ z \in \mathbb{C}\backslash\Lambda, \qquad \wp'(z)^2 = 4\ \wp(z)^3 - 60\ G_4\ \wp(z) - 140\ G_6$.

We set $g_2 = 60\ G_4$, $g_3 = 140\ G_6$, then

$$\wp'(z)^2 = 4\ \wp(z)^3 - g_2\ \wp(z) - g_3 \,. \tag{2.246}$$

Differentiating the equation above, we obtain

$$2\ \wp''(z)\ \wp'(z) = 12\ \wp'(z)\ \wp(z)^2 - g_2\ \wp'(z) \,. \tag{2.247}$$

i.e.

$$\wp''(z) = 6\ \wp(z)^2 - \frac{g_2}{2} \,. \tag{2.248}$$

Further differentiating,

$$\begin{aligned}
\wp'''(z) &= 12\ \wp'(z)\ \wp(z)\,, && (2.249)\\
\wp''''(z) &= 12\ (\wp''(z)\ \wp(z) + \wp'(z)^2)\\
&= 12\ \Big(\Big(6\ \wp(z)^2 - \frac{g_2}{2}\Big)\ \wp(z) + 4\ \wp(z)^3 - g_2\ \wp(z) - g_3\Big)\\
&= 12\ \left(10\ \wp(z)^3 - \frac{3}{2}\ g_2\ \wp(z) - g_3\right)\,, && (2.250)
\end{aligned}$$

and so on, i.e. all the derivatives of the $\wp$-function are polynomials in the $\wp$-function and its derivative $\wp'$, in agreement with the fact that every elliptic function is a rational combination of $\wp(z)$ and $\wp'(z)$.

Next, we show that the polynomial in $\wp(z)$,

$$\wp'(z)^2 = 4\ \wp(z)^3 - g_2\ \wp(z) - g_3\,, \tag{2.251}$$

has distinct roots, and so a discriminant (see Appendix G),

$$\Delta = g_2^3 - 27\ g^2 \neq 0\,. \tag{2.252}$$

Λ is periodic. Invariance under translations implies that for $2w \in \Lambda$, $\wp'(z)$ has the same value at w and $-w$. Further, since $\{\omega_1, \omega_2\}$ are a basis for Λ, we can take $\omega_3 = (\omega_1 + \omega_2)$, and since $\wp'(z)$ is odd, we have

$$\wp'\big(\frac{\omega_i}{2}\big) = -\wp'\big(-\frac{\omega_i}{2}\big) = -\wp'\big(\frac{\omega_i}{2}\big), \qquad i = 1, 2, 3\,. \tag{2.253}$$

So $\wp'\big(\frac{\omega_i}{2}\big) = 0$, and we can write

$$\wp'(z)^2 = 4\ \big(\wp(z) - \wp\big(\frac{\omega_1}{2}\big)\big)\big(\wp(z) - \wp\big(\frac{\omega_2}{2}\big)\big)\big(\wp(z) - \wp\big(\frac{\omega_3}{2}\big)\big)\,, \tag{2.254}$$

with $\wp\big(\frac{\omega_1}{2}\big) + \wp\big(\frac{\omega_2}{2}\big) + \wp\big(\frac{\omega_3}{2}\big) = 0$.

Likewise, an elliptic curve,

$$y^2 = 4\ x^3 - g_2\ x - g_3\,, \tag{2.255}$$

has distinct roots, so for $a_1 \neq a_2 \neq a_3$ it can be written as

$$y^2 = 4\ (x - a_1)(x - a_2)(x - a_3)\,, \tag{2.256}$$

with $a_1 + a_2 + a_3 = 0$.

The periods ω_1, ω_2 of the elliptic curve (2.256) take the form,

$$\omega_1 = \sqrt{a_3 - a_1} \int_{a_1}^{a_2} \frac{dx}{y} = 2\ K\Big(\sqrt{\frac{a_2 - a_1}{a_3 - a_1}}\Big)\,, \tag{2.257}$$

$$\omega_2 = \sqrt{a_3 - a_1} \int_{a_3}^{a_2} \frac{dx}{y} = 2i\ K\Big(\sqrt{\frac{a_3 - a_2}{a_3 - a_1}}\Big)\,, \tag{2.258}$$

where $K(\lambda)$ is the complete elliptic integral of the first kind (see app. F).

As we said, ω_1 and ω_2 are $\mathbb{R}$-linearly independent, so when the roots a_1, a_2, a_3 are real, ω_1 can be chosen to be real, then ω_2 must be complex, i.e. it must have an imaginary part.

2.9.1 Elliptic polylogarithms

After partial fractioning the integrand of an integral with elliptic curves in the denominator, we only need to consider integrals of the form,

$$\int \frac{dx}{y}\, x^k, \qquad \int \frac{dx}{y\,(x-c)^k}\,, \tag{2.259}$$

where $k \in \mathbb{Z}$ and c is a constant.

Integrating by parts, the integrals above can be reduced to a linear combination of

$$\int \frac{dx}{y}, \qquad \int \frac{x\,dx}{y}, \qquad \int \frac{dx}{y\,(x-c)}\,, \tag{2.260}$$

which play the same role as the master integrals.

The elliptic polylogarithms (eMPL) may be defined in terms of the coordinates (x, y) of the elliptic curve,

$$\mathrm{E}_3\!\begin{pmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{pmatrix}\!; x, \vec{a}\Big) = \int_0^x dt\; \phi_{n_1}(c_1, t)\; \mathrm{E}_3\!\begin{pmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{pmatrix}\!; t, \vec{a}\Big)\,, \tag{2.261}$$

with $n_i \in \mathbb{Z}$, $c_i \in \hat{\mathbb{C}}$, $\vec{a} = (a_1, a_2, a_3)$ is the vector of the singularities of the elliptic curve and $\mathrm{E}_3(;x, \vec{a}) = 1$. We assume here that the elliptic curve is a cubic polynomial.

Since we want elliptic polylogarithms to have at most logarithmic singularities, each ϕ_n can have at most simple poles. In particular, for $n = 0$,

$$\phi_0(x) = \frac{\sqrt{a_3 - a_1}}{2\,y} = \frac{\sqrt{a_3 - a_1}}{2\,\sqrt{(x-a_1)(x-a_2)(x-a_3)}}\,. \tag{2.262}$$

$w = \dfrac{dx}{2y}$ is the differential of the elliptic curve, it is holomorphic and non-vanishing (see App. G), thus ϕ_0 is free of poles, and $\int dx\; \phi_0$ is related to the incomplete elliptic integral of the first kind. $\phi_{\pm 1}(c, x)$ have a simple pole at $x = c$,

$$\phi_1(c, x) = \frac{1}{x-c}, \qquad \phi_{-1}(c, x) = \frac{y_c}{y\,(x-c)}\,, \tag{2.263}$$

with $y_c = (c-a_1)(c-a_2)(c-a_3)$.

ϕ_{-1} can be reduced to simpler integrals using the IBP, ϕ_1 is the integration kernel of MPLs, so

MPLs are a subset of eMPLs,

$$\mathrm{E}_3\begin{pmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{pmatrix}; x\Big) = G(c_1, \dots, c_k; x) \,. \tag{2.264}$$

The details of how the ϕ_n for $|n| \geq 2$ are chosen and constructed can be found in ref. [60].

2.9.2 Isogenies

Since elliptic curves are characterised also by a zero point, given two elliptic curves E and $E^{'}$, an isogeny is a map that preserves the zero point,

$$\phi: \; E \to E^{'} \quad \text{s.t. } \phi(0) = 0 \,. \tag{2.265}$$

If $P = (x, y)$ is a point on E: $y^2 = x^3 + A\, x + B$, an isogeny that elliptic curves have is the multiplication by n: $P \to nP$. Then the x- and y- coordinates of nP are rational functions of the x- and y-coordinates of P.

Consider two elliptic curves E and $E^{'}$ with lattices Λ and $\Lambda^{'}$. The elliptic curves have complex multiplication if there is an isogeny,

$$\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda^{'} \; / \; \phi(z) = c\; z \bmod \Lambda^{'} \,, \tag{2.266}$$

for a complex number $c \in \mathbb{C}$. Hence $c\,\Lambda \subset \Lambda^{'}$. If $c\,\Lambda = \Lambda^{'}$, then E and $E^{'}$ are isomorphic. Two isomorphic curves have the same j-invariant (see App. G), $j = j^{'}$.

E.g. for the equal-mass sunrise integral, we have found an elliptic integral with elliptic curve,

$$y^2 = (1 - x^2)(1 - k^2\, x^2) \,. \tag{2.267}$$

We could have also looked at the maximal cut. This would yield an elliptic integral with an elliptic curve given by another quartic polynomial. The two elliptic curves would be isogenic, but not isomorphic.

If E and $E^{'}$ are isomorphic, $c\,\Lambda = \Lambda^{'}$, and all the lattices with periods ω_1 and ω_2 are equivalent to a lattice with periods 1 and $\tau = \dfrac{\omega_2}{\omega_1}$. Λ and $\Lambda^{'}$ are called homothetic, τ is called fundamental lattice period or modular parameter.

2.9.3 Iterated integrals on a torus

The similarity between the elliptic curve E (2.255),

$$y^2 = 4\, x^3 - g_2\, x - g_3 = 4\, (x - a_1)(x - a_2)(x - a_3) \,, \tag{2.268}$$

over $\mathbb{C}$, and the elliptic function (2.251),

$$\wp'^2 = 4\,\wp^3 - g_2\,\wp - g_3 = 4\,(\wp - a_1)(\wp - a_2)(\wp - a_3)\,, \tag{2.269}$$

on the torus $\mathbb{C}/\Lambda$, is not a coincidence.

The uniformization theorem says that every elliptic curve over $\mathbb{C}$ is parametrised by elliptic functions, precisely that if A and B are complex number such that $\Delta = A^3 - 27\,B^2 \neq 0$, there is a unique lattice $\Lambda \subset \mathbb{C}$, such that $g_2(\Lambda) = A$ and $g_3(\Lambda) = B$. The proof uses the theory of modular functions, i.e. functions whose domain is the set of lattices in $\mathbb{C}$ (which we will only mention briefly).

A consequence of the uniformization theorem is that there is a complex analytic isomorphism,

$$F:\ E(\mathbb{C}) \to \mathbb{C}/\Lambda\ /\ F(P) = \int_O^P \frac{dx}{y} \bmod \Lambda\,, \tag{2.270}$$

in particular the differential $\frac{dx}{y}$ on E "pulls back" to $\frac{d\wp(z)}{\wp'(z)} = dz$ on $\mathbb{C}/\Lambda$, and to a point $[x, y, 1]$ on E is associated a point $z = [\wp(z), \wp'(z), 1]$ on the torus $\mathbb{C}/\Lambda$,

$$[x, y, 1] \to [\wp(z), \wp'(z), 1]\,. \tag{2.271}$$

If $y = 0$, then $x = a_i$. Correspondingly, on the torus $\mathbb{C}/\Lambda$, we have $\wp'(z) = 0$, when $z = \frac{\omega_i}{2}$, i.e. on the half-periods, so

$$[a_i, 0, 1] \to \left[\wp\left(\frac{\omega_i}{2}\right), 0, 1\right]\,. \tag{2.272}$$

If $y \neq 0$, then $F(z) = \wp(z) - x$ is an elliptic function, with a double pole at $z = 0$ on $\mathbb{C}/\Lambda$. Since an elliptic function has equal numbers of zeros and poles, then $F(z)$ has two zeros or a double zero on $\mathbb{C}/\Lambda$, thus $\wp(z) = x$ has two solutions, which differ by a sign since $\wp(z)$ is even.

Since every elliptic function is a rational combination of $\wp(z)$ and $\wp'(z)$, if $R(x, y)$ is a rational function of two variables, $R(x, y)$ on E is mapped to $R((\wp(z), \wp'(z)))$ on $\mathbb{C}/\Lambda$ and vice-versa, so the field of elliptic function on the torus is isomorphic to the field of rational functions on elliptic curves.

An abelian differential on E is a differential form of type $dx\ R(x, y)$. There are three types of abelian differentials: those of the first kind are holomorphic on E; those of the second kind are meromorphic, but the residue at every point must vanish; those of the third kind are meromorphic, with non-vanishing residues. Now, we examine how abelian differentials on E are mapped to differential forms on the torus $\mathbb{C}/\Lambda$.

We already know that $w = \frac{dx}{2y}$ is holomorphic and non-vanishing on E, so it is of the first kind, and it is mapped to the holomorphic differential $\frac{d\wp(z)}{\wp'(z)} = dz$ on $\mathbb{C}/\Lambda$.

$\frac{x\ dx}{y}$ has a double pole without residue at infinity, so it is of the second kind, and on $\mathbb{C}/\Lambda$ it corresponds to $\wp(z)\ dz = \frac{dz}{z^2} + \mathcal{O}(z^0)$.

$\dfrac{dx}{y\,(x-c)}$ has a simple pole at $x=c$, with residue $y=\sqrt{f(x_c)}$, so it is of the third kind.

In general, an abelian differential on E is mapped to the differential form $f(z)\,dz$ on $\mathbb{C}/\Lambda$, with $f(z)$ an elliptic function. However, $f(z)$ has at least two poles, so an abelian differential cannot have a simple pole. This is related to the fact that the primitive F of an elliptic function $F(z)=\int_{z_0}^{z} dt\, f(t)$ is not periodic, because the lower integration limit z_0 breaks the invariance under translations by periods. So

$$F(z+\omega)=\int_{z_0}^{z+\omega} dt\; f(t)=F(z)+C\,, \tag{2.273}$$

where C is a constant (with respect to z) that may depend on ω,

$$C=\int_{z}^{z+\omega} dt\; f(t)=\int_0^{\omega} dt\; f(t)\,. \tag{2.274}$$

$F(z)$ is called quasi-periodic. An example of quasi-periodic function is the Weierstrass ζ function,

$$\begin{aligned}\zeta(z) &= \frac{1}{z}-\int_0^z dt\;\big(\wp(t)-\frac{1}{t^2}\big)\\ &= \frac{1}{z}-\int_0^z dt\sum_{\omega\in\Lambda,\omega\neq 0}\big(\frac{1}{(t-\omega)^2}-\frac{1}{\omega^2}\big)\\ &= \frac{1}{z}+\sum_{\omega\in\Lambda,\omega\neq 0}\big(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\big)\,. \end{aligned}\tag{2.275}$$

Note that

$$\frac{d\zeta(z)}{dz}=-\frac{1}{z^2}-\sum_{\omega\in\Lambda,\omega\neq 0}\big(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\big)=-\wp(z)\,, \tag{2.276}$$

so the Weierstrass ζ function is a primitive, up to a sign, of the $\wp$-function. The ζ function is not invariant under the translations, $\zeta(z+\omega_i)=\zeta(z)+2\,\zeta(\frac{\omega_i}{2})$, but it has a simple pole at $z=0$, so it can be used to construct differential forms with simple poles on $\mathbb{C}/\Lambda$.

This allows us to introduce another definition of elliptic polylogarithms, which are called iterated integrals of modular forms,

$$\tilde{\Gamma}\Big(\begin{matrix} n_1 & \dots & n_k\\ z_1 & \dots & z_k\end{matrix};z,\tau\Big)=\int_0^z dt\; g^{(n_1)}(t-z_1;\tau)\;\tilde{\Gamma}\Big(\begin{matrix} n_2 & \dots & n_k\\ z_2 & \dots & z_k\end{matrix};t,\tau\Big)\,. \tag{2.277}$$

Firstly, let us describe the parameter τ. As we said, lattices with periods ω_1 and ω_2 are isomorphic to a lattice with periods 1 and $\tau=\omega_2/\omega_1$, with τ the fundamental lattice period or modular parameter, ω_1 and ω_2 are $\mathbb{R}$-linearly independent, and we may choose $\omega_1\in\mathbb{R}^+$, and ω_2 with a positive imaginary part. Thus $\mathrm{Im}(\tau)>0$. So, introducing the upper half plane, $\mathbb{H}=\{\tau\in\mathbb{C};\ \mathrm{Im}(\tau)>0\}$, the isomorphism class of lattices is labelled by

$$\Lambda_\tau=\{n_1+n_2\tau\ /\ n_1,n_2\in\mathbb{Z},\ \tau\in\mathbb{H}\}\,. \tag{2.278}$$

On the upper half plane $\mathbb{H}$ acts the special linear group $\mathrm{SL}(2,\mathbb{R})$, by Möbius transformations,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{H} \to \mathbb{H}, \quad z \to \gamma(z) = \frac{a\,z + b}{c\,z + d}\,, \tag{2.279}$$

with $\det(g) = 1$.

$\mathbb{H}$ is mapped to $\mathbb{H}$ because

$$\mathrm{Im}\big(\gamma(z)\big) = \frac{\mathrm{Im}(z)}{|c\,z + d|^2}\,, \qquad \text{(please check)}\,. \tag{2.280}$$

γ and $-\gamma$ act in the same way on $\mathbb{H}$, so we take $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$, with $\mathbb{Z}_2 = \{\mathbb{1}, -\mathbb{1}\}$. Different values of τ may correspond to the same elliptic curve. They do if and only if they are related by a $\mathrm{PSL}(2,\mathbb{Z})$ modular transformation, since $\Gamma_1 \equiv \mathrm{SL}(2,\mathbb{Z})$ is called modular group. It is called so because the points of $\Gamma_1/\mathbb{H}$ are moduli, i.e. parameters, for the isomorphism class of lattices. So $\Gamma_1/\mathbb{H}$ is a moduli space.

A meromorphic function $f : \Gamma_1/\mathbb{H} \to \mathbb{C}$, invariant under Γ_1, $f(\gamma z) = f(z)$, is called a modular function. The function must be meromorphic because there are no holomorphic functions on $\mathbb{C}/\Lambda$. For this reason, one introduces modular forms, which are holomorphic functions on $\mathbb{H}$, such that the modular function is the quotient of two modular forms. A modular form of weight k transforms as

$$f\big(\frac{az + b}{cz + d}\big) = (cz + d)^k\; f(z)\,, \qquad \forall\; \gamma \in \Gamma_1\,. \tag{2.281}$$

An example of modular forms is the Eisenstein series G_{2k} of weight $2k$, for $k > 1$ (for $k = 1$, it is a quasi-modular form).

In the definition of the eMPLs through the $\tilde{\Gamma}$ functions (2.277), the integration kernels $g^{(n_1)}(t - z_1; \tau)$ are modular forms...

Appendix A

n-gluon scattering

Let us consider the scattering of $2 \to (n-2)$ gluons,

$$g(-p_1) + g(-p_n) \to g(p_2) + \ldots + g(p_{n-1})\,. \tag{A.1}$$

We parametrise the momenta in light-cone coordinates,

$$\begin{aligned} &p_1 = (p_i^+, 0; 0, 0) \qquad p_n = (0, p_n^-; 0, 0)\,, \\ &p_i = (|p_{i\perp}|e^{y_i}, |p_{i\perp}|e^{-y_i}; p_{ix}, p_{iy}), \qquad i = 2, \ldots, n-1\,, \\ &p_{i_\perp} = p_{ix} + ip_{iy}, \qquad y_i \text{ is the rapidity of the i}^{\text{th}} \text{ gluon}\,. \end{aligned} \tag{A.2}$$

Momentum conservation is

$$-p_1^+ = \sum_{i=2}^{n-1} p_i^+; \qquad -p_n^- = \sum_{i=2}^{n-1} p_i^-; \qquad 0 = \sum_{i=2}^{n-1} p_{i_\perp}\,. \tag{A.3}$$

From eq. (1.25), for $p_i^+ \neq 0$, the Weyl spinors are

$$\xi_+(p_i) = \frac{1}{\sqrt{p_i^+}} \begin{pmatrix} p_i^+ \\ p_{i\perp} \end{pmatrix}, \qquad \xi_-(p_i) = \frac{1}{\sqrt{p_i^+}} \begin{pmatrix} -p_{i\perp}^* \\ p_i^+ \end{pmatrix}, \qquad i = 2, ..., n-1\,. \tag{A.4}$$

From eq. (1.24), for $p^+ = 0$, we use

$$\xi_+(p) = \begin{pmatrix} 0 \\ \sqrt{p^-} \end{pmatrix}, \qquad \xi_-(p) = -\begin{pmatrix} \sqrt{p^-} \\ 0 \end{pmatrix}. \tag{A.5}$$

After analytic continuation, the negative-energy gluons have spinors[1]

$$\xi_+(p_1) = i \begin{pmatrix} \sqrt{-p_1^+} \\ 0 \end{pmatrix}; \qquad \xi_-(p_1) = i \begin{pmatrix} 0 \\ \sqrt{-p_1^+} \end{pmatrix},$$
$$\xi_+(p_n) = i \begin{pmatrix} 0 \\ \sqrt{-p_n^-} \end{pmatrix}; \qquad \xi_-(p_n) = -i \begin{pmatrix} \sqrt{-p_n^-} \\ 0 \end{pmatrix}. \tag{A.6}$$

The spinor products are

$$\langle ij\rangle = \xi_-^\dagger(p_i)\xi_+(p_j) = -p_{i\perp}\sqrt{\frac{p_j^+}{p_i^+}} + p_{j\perp}\sqrt{\frac{p_i^+}{p_j^+}} \qquad i,j = 2,\ldots,n-1\,, \tag{A.7}$$
$$\langle 1k\rangle = \xi_-^\dagger(p_1)\xi_+(p_k) = ip_{k\perp}\sqrt{\frac{-p_1^+}{p_k^+}} \qquad k = 2,\ldots,n-1\,, \tag{A.8}$$
$$\langle kn\rangle = \xi_-^\dagger(p_k)\xi_+(p_n) = i\sqrt{-p_n^- p_k^+} \qquad k = 2,\ldots,n-1\,, \tag{A.9}$$
$$\langle 1n\rangle = \xi_-^\dagger(p_1)\xi_+(p_n) = -\sqrt{(-p_1^+)(-p_n^-)} = -\sqrt{s}\,. \tag{A.10}$$

Note that the spinor products with negative-energy gluons acquire as expected a sign factor when complex conjugated, $[pk] = \text{sign}(p^0k^0)\langle kp\rangle^*$.

In fact,

$$[k1] = \xi_+^\dagger(p_k)\xi_-(p_1) = ip_{k\perp}^*\sqrt{\frac{-p_1^+}{p_k^+}} = -\langle 1k\rangle^*\,, \tag{A.11}$$
$$[nk] = \xi_+^\dagger(p_n)\xi_-(p_k) = i\sqrt{-p_n^- p_k^+} = -\langle kn\rangle^*\,, \tag{A.12}$$
$$[n1] = \xi_+^\dagger(p_n)\xi_-(p_1) = -\sqrt{(-p_1^+)(-p_n^-)} = -\sqrt{s} = \langle 1n\rangle^*\,. \tag{A.13}$$

A.1 Multi-Regge Kinematics

The multi-Regge kinematics (MRK) is defined by a strong ordering in rapidity of the outgoing gluons,

$$y_2 >> y_3 >> \ldots >> y_{n-1}\,, \tag{A.14}$$

with comparable transverse momenta

$$|p_{\perp 2}| \simeq \ldots \simeq \left|p_{\perp(n-1)}\right|. \tag{A.15}$$

In light-cone coordinates, that is equivalent to the strong ordering,

$$p_2^+ >> \ldots >> p_{n-1}^+\,, \qquad p_2^- << \ldots << p_{n-1}^-\,. \tag{A.16}$$

[1] The analytic continuation to negative energy of the incoming gluons is done after any possible complex conjugation, so e.g. $\xi_+^\dagger(p_1) = i(\sqrt{-p_1^+}, 0)$.

The leading contribution to momentum conservation becomes

$$-p_1^+ \simeq p_2^+ \,, \qquad -p_n^- \simeq p_{n-1}^- \,. \tag{A.17}$$

Likewise, the leading contribution to the spinor product is

$$\langle ij \rangle \simeq p_{j\perp} \sqrt{\frac{p_i^+}{p_j^+}}, \qquad \text{for} \quad p_i^+ >> p_j^+ \,. \tag{A.18}$$

The spinor product $\langle 1k \rangle$, $\langle kn \rangle$, $\langle 1n \rangle$ are formally the same, however in computing them one retains the leading contribution to the momentum conservation, e.g.

$$\begin{aligned} \langle kn \rangle &= i\sqrt{-p_n^- p_k^+} \simeq i\sqrt{p_{n-1}^- p_k^+} \qquad k = 2, \ldots, n-1 \,, \\ \langle 1n \rangle &= -\sqrt{(-p_1^+)(-p_n^-)} = -\sqrt{s} \simeq \sqrt{p_2^+ p_{n-1}^-} \,. \end{aligned} \tag{A.19}$$

Appendix B

Basics of projective space

Let us consider a polynomial function of two variables, $f(x,z)$, which is homogeneous of degree d,

$$f(\lambda x, \lambda z) = \lambda^d f(x,z)\,. \tag{B.1}$$

Note that any function of one variable $g(x)$ of total degree d can be made homogeneous by adding one variable,

$$g(x,z) = z^d\; g\big(\frac{x}{z}\big)\,, \tag{B.2}$$

e.g.

$$g(x) = x^3 + x \to g(x,z) = z^3\big(\big(\frac{x}{z}\big)^3 + \frac{x}{z}\big) = x^3 + xz^2\,. \tag{B.3}$$

Of course, for a homogeneous polynomial $f(x,z)$, if x_0 is a root also λx_0 is a root. Thus, we would like to divide out by all solutions related by simple rescaling.

We say that $(x_1, \dots, x_{n+1})$ and $(x_1^{'}, \dots, x_{n+1}^{'})$ are equivalent if there is a $\lambda \in \mathbb{K}$, such that $x_i = \lambda x_i^{'}$, for $i = 1, \dots, n+1$. The equivalence class is $[x_1 : \ldots : x_{n+1}]$. The projective space is

$$\mathbb{P}^n(\mathbb{K}) = \{[x_1 : \ldots : x_{n+1}] \in \mathbb{K}^{n+1}/ \text{ not all } x_i = 0\}\,. \tag{B.4}$$

Then the solutions of a homogeneous polynomial $f(x_1, \dots, x_{n+1})$ belong to the projective space $\mathbb{P}^n(\mathbb{K})$,

$$S := \{[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n(\mathbb{K})/\; f(x_1, \dots, x_{n+1}) = 0\}\,. \tag{B.5}$$

If $x_{n+1} = 0$, we get the point at infinity.

In sec. 1.15, $\mathbb{K} = \mathbb{C}$ and $n = 1$, and $\mathbb{P}^1(\mathbb{C})$ is $\mathbb{CP}^1$.

Appendix C

Pfaffian

Consider a $2n \times 2n$ antisymmetric matrix Ψ, and the set S of partitions of $\{1, 2, \ldots, 2n\}$ into pairs. The elements $a \in S$ can be written as

$$a = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}\,, \qquad i_k < j_k\,, \qquad i_1 < i_2 < \ldots < i_n\,. \tag{C.1}$$

There are $\dfrac{(2n)!}{2^n\; n!}$ partitions.

Then we consider the corresponding permutations,

$$\sigma_a = \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \ldots & i_n & j_n \end{Bmatrix}\,, \tag{C.2}$$

and the product,

$$\psi_a = \mathrm{sgn}(\sigma_a)\; a_{i_1 j_1}\; a_{i_2 j_2} \cdots a_{i_n j_n}\,. \tag{C.3}$$

Then

$$\mathrm{Pf}(\Psi) = \sum_{a \in S}\; \psi_a\,. \tag{C.4}$$

Note that

$$\mathrm{Pf}(\Psi)^2 = \det(\Psi)\,. \tag{C.5}$$

E.g. suppose that $n = 2$,

$$\Psi = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}\,. \tag{C.6}$$

There are $\dfrac{4!}{4 \cdot 2!} = 3$ partitions, given by

pairs	(1,2)(3,4)	(1,3)(2,4)	(1,4)(2,3)
signatures	$+$	$-$	$+$

The Pfaffian is given by

$$\mathrm{Pf}(\Psi) = a_{12}\ a_{34} - a_{13}\ a_{24} + a_{14}\ a_{23}\,. \tag{C.7}$$

Appendix D

One-loop colour decomposition

Since we are going to analyse one-loop n-gluon amplitudes in some detail, let us display their colour decomposition.

One-loop n-gluon amplitudes may feature a gluon loop or quark loop with n_f flavours of quarks. Their trace-based colour decomposition is

$$\begin{aligned} M_n^{(1)}(1,\ldots,n) &= g^n\big[N_C \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}})A_{n;1}^{[1]}(\sigma_1\cdots\sigma_n) \\ &+ \sum_{c=2}^{[n/2]+1}\sum_{\sigma\in S_n/S_{n;c}} \mathrm{tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_{c-1}}})\mathrm{tr}(T^{a_{\sigma_c}}\cdots T^{a_{\sigma_n}})A_{n;c}(\sigma_1\cdots\sigma_n) \\ &+ n_f \sum_{\sigma_n\in S_n/\mathbb{Z}_n} \mathrm{tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}})A_{n;1}^{[1/2]}(\sigma_1\cdots\sigma_n)\big]\,. \end{aligned} \tag{D.1}$$

where $S_{n;c}$ is the subset of S_n that leaves the double trace invariant, and $[x]$ is the greatest integer less than or equal to x. The superscript $[j]$ with $j = 1, 1/2$ denotes the spin circulating in the loop. $A_{n;1}^{[1]}$ and $A_{n;1}^{[1/2]}$ are coloured-ordered. $A_{n;1}^{[1]}$ yields the leading contribution in the large-N_c limit. The sub-leading amplitudes $A_{n;c}$ can be written in terms of linear combinations of $A_{n;1}^{[1]}$ [43].

The multiperipheral colour decomposition is [18]

$$\begin{aligned} M_n^{(1)}(1,\ldots,n) &= g^n\big[N_C \sum_{\sigma\in S_n/\mathbb{Z}_n\otimes R} \mathrm{tr}(F^{a_{\sigma_1}}\cdots F^{a_{\sigma_n}})A_{n;1}^{[1]}(\sigma_1\cdots\sigma_n) \\ &+ n_f \sum_{\sigma_n\in S_n/\mathbb{Z}_n} \mathrm{tr}(T^{a_{\sigma_1}}\cdots T^{a_{\sigma_n}})A_{n;1}^{[1/2]}(\sigma_1\cdots\sigma_n)\big]\,. \end{aligned} \tag{D.2}$$

where R is the reflection, $R(1,\ldots,n) = (n,\ldots,1)$, thus the independent sub-amplitudes $A_{n;1}^{[1]}$ are $\frac{(n-1)!}{2}$.

Appendix E

Integration-by-part identities

Let us suppose that every Feynman integral with momenta in the numerator has been reduced to scalar integrals, by expressing e.g. every scalar product by a difference of propagators, e.g. for $p^2 = 0$

$$\int d^d\ell \; \frac{\ell \cdot p}{\ell^2(\ell+p)^2} = \int d^d\ell \; \frac{1}{\ell^2(\ell+p)^2} \; \frac{1}{2} \; [(\ell+p)^2 - \ell^2] \, . \tag{E.1}$$

The integration-by-part (IBP) identities, i.e. the reduction of scalar integrals to a linearly independent set of scalar integrals, stem from using the divergence theorem on the integrals. For example, let us take the massive tadpole $\int d^d\ell \; \frac{\ell^\mu}{D}$, with $D = \ell^2 - m^2$. Then the divergence theorem states that

$$\int_M d^d\ell \; \frac{\partial}{\partial \ell^\mu} \; \frac{\ell^\mu}{D} = \int_{\partial M} ds^\mu \; \frac{\ell^\mu}{D} = 0 \, , \tag{E.2}$$

where M is the space of integration, and ∂M its boundary (usually the $(D-1)$-sphere). But

$$\begin{aligned} 0 \;&=\; \int d^d\ell \; \frac{\partial}{\partial \ell^\mu} \; \frac{\ell^\mu}{\ell^2 - m^2} = \int d^d\ell \; \Big(\frac{d}{\ell^2 - m^2} - 2 \; \frac{\ell^2}{(\ell^2 - m^2)^2} \Big) \\ &=\; \int d^d\ell \; \Big(\frac{d}{D} - 2 \; \frac{D + m^2}{D^2} \Big) = (d-2) \; I(1) - 2m^2 I(2) \, , \end{aligned} \tag{E.3}$$

where $I(n) = \int d^d\ell \frac{1}{D^n}$, so we get the IBP,

$$(d-2) \; I(1) - 2 \; m^2 \; I(2) = 0 \, , \tag{E.4}$$

and we can express $I(2)$ as a function of $I(1)$, i.e. in the case of the massive tadpole, we have only one master integral.

Appendix F

Complete elliptic integrals

Considering an elliptic curve in Jacobi form,

$$y^2 = (1-x^2)(1-k^2x^2)\,, \tag{F.1}$$

the complete elliptic integral of the first kind is

$$\begin{aligned} K(k) = \int_0^1 \frac{dx}{y} &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\,, \end{aligned} \tag{F.2}$$

the complete elliptic integral of the second kind is

$$\begin{aligned} E(k) = \int_0^1 dx\ \frac{1-k^2x^2}{y} = \int_0^1 dx\ \frac{y}{1-x^2} &= \int_0^1 dx\ \sqrt{\frac{1-k^2x^2}{1-x^2}} \\ &= \int_0^{\pi/2} d\theta\ \sqrt{1-k^2\sin^2\theta}\,, \end{aligned} \tag{F.3}$$

and is related, as we saw in (2.227), to the perimeter I of an ellipse with radii a and b, and $b > a$,

$$I = 4b\ E\Big(\sqrt{1-\frac{a^2}{b^2}}\Big)\,.$$

The incomplete elliptic integrals are obtained from the complete ones by leaving the upper limit of integration undetermined, i.e. by replacing 1 by x, or $\pi/2$ by θ.

Appendix G

Weierstrass form of an elliptic curve

As discussed in the App. B, a projective curve in the projective plane $\mathbb{P}^n$ is the set of solutions,

$$C := \{[X_1 : \ldots : X_{n+1}] \in \mathbb{P}^n \ / \ F(X_1, \ldots, X_{n+1}) = 0\} \,. \tag{G.1}$$

of a homogeneous polynomial $F(X_1, \ldots, X_{n+1})$.

We introduce the non-homogeneous polynomial,

$$f(x_1, \ldots, x_n) = F(X_1, \ldots, X_n, 1) \,. \tag{G.2}$$

The curve $f(x_1, \ldots, x_n) = 0$ is the affine part of the projective curve C. The point $[X_1 : \ldots : X_{n+1}]$ with $X_{n+1} = 0$ describes the point at infinity. It can be shown that C can be written as the union of its affine curve and its point at infinity.

An elliptic curve in Weierstrass form is a curve in $\mathbb{P}^2$ with an equation of the form,

$$Y^2 Z + a_1\, XYZ + a_3\, YZ^2 = X^3 + a_2\, X^2 Z + a_4\, XZ^2 + a_6\ Z^3 \,, \tag{G.3}$$

with a base point $\mathbb{O} = [0, 1, 0]$. Its affine curve is

$$E: \ y^2 + a_1\, xy + a_3\, y = x^3 + a_2\, x^2 + a_4\, x + a_6 \,, \tag{G.4}$$

with the extra point $\mathbb{O} = [0, 1, 0]$ at infinity, is called the long Weierstrass form. We can simplify it through the substitution,

$$y \to \frac{1}{2}\, (y - a_1\, x - a_3) \,. \tag{G.5}$$

Then we get (please check)

$$E: \ y^2 = 4\, x^3 + b_2\, x^2 + 2\, b_4\, x + b_6 \,, \tag{G.6}$$

with $b_2 = a_1^2 + 4\, a_2, \quad b_4 = 2\, a_4 + a_1 a_3, \quad b_6 = a_3^2 + 4\, a_6$.

Substituting further $(x,y) \to (\frac{x - 3\,b_2}{36}, \frac{y}{108})$, we obtain the equation,

$$y^2 = x^3 - 27\,c_4\,x - 54\,c_6\,, \tag{G.7}$$

with $c_4 = b_2^2 - 24\,b_4, \quad c_6 = -b_2^3 + 36\,b_2\,b_4 - 216\,b_6$.

If an elliptic curve has the form,

$$y^2 = x^3 + A\,x + B\,, \tag{G.8}$$

it is in the short Weierstrass form.

Here are some examples of elliptic curves, which have distinct roots.

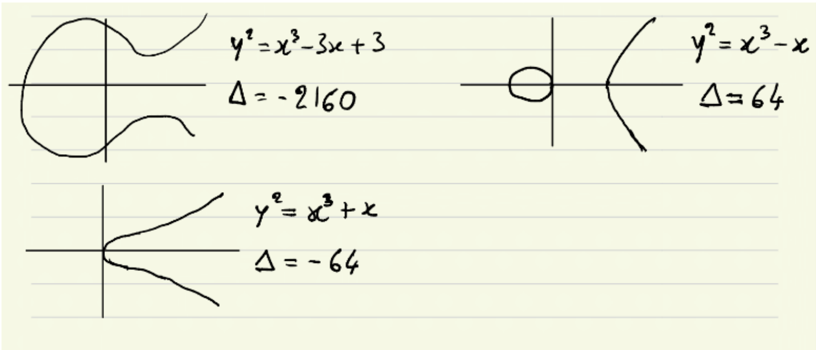


Figure G.1: Examples of elliptic curves with distinct roots: On the left panels with one real root; on the right panel with three real roots.

For a polynomial equation, $f(X_1, \ldots, X_n) = 0$, a singular point $P = (X_1^0, \ldots, X_n^0)$ is characterised by

$$\frac{\partial f(P)}{\partial X_1} = \ldots = \frac{\partial f(P)}{\partial X_n} = 0\,. \tag{G.9}$$

For $f(x,y) = y^2 - f(x) = 0$, with $f(x) = x^3 + A\;x + B$,

$$\begin{cases} \left.\dfrac{\partial f}{\partial y}\right|_P = 2\,y_0 = 0 \quad \Rightarrow \quad f(x_0) = 0 \\ \left.\dfrac{\partial f}{\partial x}\right|_P = f^{'}(x_0) = 0 \end{cases} \tag{G.10}$$

so $f(x)$ and $f^{'}(x)$ have the common singularity x_0, i.e. x_0 is a double singularity of f. Examples of singular cubic curves with a double singularity,

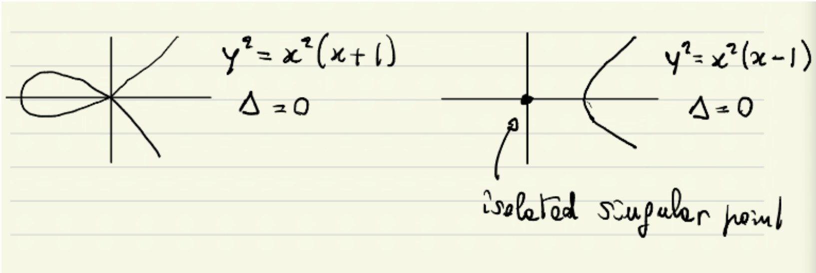


Figure G.2: Examples of singular cubic curves with double singularities.

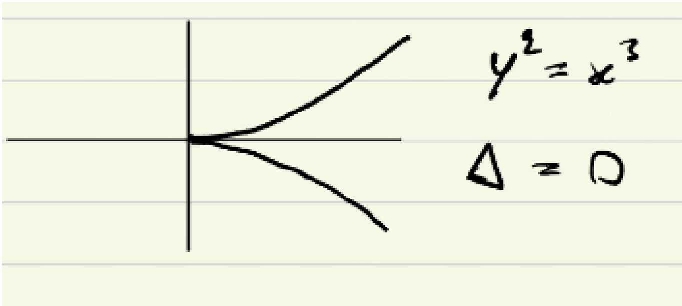


Figure G.3: Example of singular cubic curve with triple singularity.

We introduce the quantities:

For $y^2 = x^3 + A\,x + B$,

$$\Delta = -16(4\,A^3 + 27\,B^2), \qquad j = -1728\,\frac{(4A)^3}{\Delta}\,. \tag{G.11}$$

More in general, for $y^2 = 4x^3 + b_2x^2 + 2\,]b_4\,x + b_6$, as in eq. (G.6), fixing $4b_8 = b_2b_6 - b_4^2$,

$$\Delta = -b_2^2b_8 - 8\,b_4^3 - 27\,b_6^2 + 9\,b_2b_4b_6, \quad j = \frac{(b_2^2 - 24\,b_4)^3}{\Delta}\,. \tag{G.12}$$

So for $y^2 = 4\,x^3 - g_2\,x - g_3$,

$$\Delta = g_2^3 - 27\,g_3^2, \qquad j = \frac{1728\,g_2^3}{\Delta}\,. \tag{G.13}$$

Δ is the discriminant of the Weierstrass equation. When $\Delta = 0$, the curve is singular,

$$\text{If } \Delta = 0 \text{ and } \begin{cases} A \neq 0, & \text{the curve has a node} \\ A = 0, & \text{the curve has a cusp}\,. \end{cases} \tag{G.14}$$

The only change of variables that preserves the short form of the equation is

$$x = u^2\,x', \qquad y = u^3\,y'\,, \tag{G.15}$$

with $u^4\,A' = A$ and $u^6\,B' = B$, which implies that $u'^2\,\Delta' = \Delta$ and $j' = j$, so j is the j-invariant of the elliptic curve, i.e. it is the invariant of the isomorphism class of the curve. We introduce the differential of an elliptic curve E (i.e. with no singular points),

$$\omega = \frac{d(x-x_0)}{\partial_y f(x,y)} = -\frac{d(y-y_0)}{\partial_x f(x,y)}, \quad \text{for a point } P=(x_0,y_0)\,. \tag{G.16}$$

ω is holomorphic and non-vanishing, i.e. it has no zeros and no poles (a pole of ω would imply $\partial_x f(x,y) = \partial_y f(x,y) = 0$, which would mean that E is singular at P).

For $f(x,y) = y^2 - f(x) = 0$, with $f(x) = x^3 + A\,x + B$,

$$\omega = \frac{d(x-x_0)}{2\,y}\,. \tag{G.17}$$

Appendix H

Exercises

H.1 Charge conjugation

The charge conjugation maps a fermion of a given spin into the antifermion of the same spin. The charge conjugation matrix C is defined by

$$C\gamma_\mu^* C^{-1} = -\gamma_\mu \qquad \text{with} \qquad C = C^{-1} = C^\dagger = C^T \,, \tag{H.1}$$

and its action on the Dirac spinor is $Cu_\pm = u_\mp^*$.

1. Show that $C = \pm i\gamma_2$ and choose $C = -i\gamma_2$ for further computation.

   Solution. We have

$$\gamma^{0\dagger} = \gamma^0 \,, \quad \gamma^{i\dagger} = -\gamma^i \,, \quad i = 1, 2, 3 \,, \tag{H.2}$$

   thus

$$\gamma^{1*} = \gamma^1 \,, \quad \gamma^{2*} = -\gamma^2 \,, \quad \gamma^{3*} = \gamma^3 \,. \tag{H.3}$$

   $C\gamma_\mu^* = -\gamma_\mu C$ implies that $C = e^{i\phi}\gamma_2$ and then

$$C = C^{-1} = C^\dagger = C^T \Rightarrow e^{i\phi} = \pm i \,. \tag{H.4}$$

2. What do $C\gamma_\mu^* C^{-1} = -\gamma_\mu$ and $Cu_\pm = u_\mp^*$ imply in the 2-dimensional spinors and matrices ?

   Solution.

$$Cu_+ = -i\gamma_2 u_+ = u_-^* \quad \text{gives} \quad -i\sigma_2\xi_+ = \xi_-^* \,. \tag{H.5}$$

   $C\gamma_\mu^* C^{-1} = -\gamma_\mu$ is explicitly

$$-i\gamma_2\gamma_\mu^*(-i\gamma_2) = -\gamma_\mu \Rightarrow \gamma_2\gamma_\mu^*\gamma_2 = \gamma_\mu \,, \tag{H.6}$$

which yields the relation

$$\sigma_2 \bar{\sigma}_\mu^* \sigma_2 = \sigma_\mu \ . \tag{H.7}$$

Transposing it yields

$$\sigma_2 \bar{\sigma}_\mu \sigma_2 = \sigma_\mu^T \ , \tag{H.8}$$

since $\sigma_\mu^\dagger = \sigma_\mu$ and $\sigma_2^T = -\sigma_2$.

H.2 Little group

Show that the little group of a 4-vector p^μ with $p^2 = 0$ is $ISO(2)$, the group of rotations and translations in 2 dimensions.

Solution. We have a reference vector $\hat{p}^\mu = \omega_0(1,0,0,1)$ with $\omega_0 > 0$ and we want to determine $G_{\hat{p}} = \{\Lambda : \Lambda\hat{p} = \hat{p}\}$. Using the infinitesimal Lorentz transformation,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \ , \tag{H.9}$$

we get the linear system

$$\omega^\mu{}_\nu \hat{p}^\nu = 0, \qquad \omega^{\mu\nu} = -\omega^{\nu\mu} \ , \tag{H.10}$$

of which the solution gives

$$\omega^0{}_3 = \omega^3{}_0 = 0 \qquad\qquad \omega^1{}_0 = -\omega^1{}_3 \qquad\qquad \omega^2{}_0 = -\omega^2{}_3. \tag{H.11}$$

This reduces the number of independent $\omega^{\mu\nu}$ from six to three so that

$$\frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} = \omega^{12} \underbrace{M_{12}}_{J_3} + \omega^{01} \underbrace{(M_{01} + M_{31})}_{E_1} + \omega^{02} \underbrace{(M_{02} + M_{32})}_{E_2} \ , \tag{H.12}$$

where the $M_{\mu\nu}$ are the usual generators of the Lorentz group. The commutators are

$$[J_3, E_1] = iE_2, \qquad\qquad [J_3, E_2] = -iE_1, \qquad\qquad [E_1, E_2] = 0 \ , \tag{H.13}$$

and the unitary operator corresponding to finite group elements of $G_{\hat{p}}$ is

$$S(a_1, a_2)R(\theta) = e^{-i(a_1 E_1 + a_2 E_2)} e^{-i\Theta J_3} \ . \tag{H.14}$$

The key is to notice that

$$e^{-i\Theta J_3}(a_1E_1+a_2E_2)e^{i\theta J_3}=a_1^\Theta E_1+a_2^\Theta E_2, \qquad \begin{pmatrix}a_1^\Theta\\ a_2^\Theta\end{pmatrix}=\begin{pmatrix}\cos\Theta & -\sin\Theta\\ \sin\Theta & \cos\Theta\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}, \tag{H.15}$$

which is

$$R(\Theta)S(a_1,a_2)R^{-1}(\Theta)=S(a_1^\Theta,a_2^\Theta)\Leftrightarrow R(\Theta)S(a_1,a_2)=S(a_1^\Theta,a_2^\Theta)R(\Theta)\,. \tag{H.16}$$

Now we can look at

$$S_(a_1',a_2')R(\Theta')S(a_1,a_2)R(\Theta)=S(a_1',a_2')S(a_1^{\Theta'},a_2^{\Theta'})R(\Theta')R(\Theta) \tag{H.17}$$

$$=S(a_1'+a_1^{\Theta'},a_2'+a_2^{\Theta'})R(\Theta'+\Theta)\,, \tag{H.18}$$

which of course is nothing else than the group multiplication rule,

$$(\Theta',a_1',a_2')(\Theta,a_1,a_2)=(\Theta'+\Theta,a_1'+a_1^{\Theta'},a_2'+a_2^{\Theta'})\,, \tag{H.19}$$

which we recognize as the group of rotations and translations in a two-dimensional space $ISO(2)$ and we have $G_{\hat{p}}\simeq ISO(2)$.

For further reading, see [16], chapter 2.

H.3 Spinor identities

1. Using the explicit form of $(\sigma^\mu)^{\dot{a}a}$, check that

$$\left\langle p^+\right|\gamma^\mu\left|p^+\right\rangle=\tilde{\lambda}_{\dot{a}}(p)\,(\sigma^\mu)^{\dot{a}a}\,\lambda_a(p)=2p^\mu\,, \tag{H.20}$$

i.e. the Gordon identity and likewise

$$\left\langle p^-\right|\gamma^\mu\left|p^-\right\rangle=\lambda^a(p)\,(\bar{\sigma}^\mu)_{a\dot{a}}\,\tilde{\lambda}^{\dot{a}}(p)=2p^\mu. \tag{H.21}$$

Solution. We have the identity,

$$A=\frac{1}{2}\,\mathrm{tr}(\sigma^\mu A)\bar{\sigma}_\mu\,, \tag{H.22}$$

for all complex 2×2 matrices. In particular, we have

$$x^\mu\to x_{a\dot{a}}=(x^\mu\bar{\sigma}_\mu)_{a\dot{a}}\,, \tag{H.23}$$

but also from (H.22)

$$x=\frac{1}{2}\,\mathrm{tr}(\sigma^\mu x)\bar{\sigma}_\mu\,. \tag{H.24}$$

Comparing the coefficients in front of $\bar{\sigma}$ immediately shows these identities.

2. Use the identity $(\sigma^\mu)^{\dot{a}a} = -\epsilon^{\dot{a}\dot{b}}\,(\bar{\sigma}^\mu)^T_{\dot{b}b}\,\epsilon^{ba}$ to show that

$$\lambda^a(p)\,(\bar{\sigma}^\mu)_{a\dot{a}}\,\tilde{\lambda}^{\dot{a}}(q) = \tilde{\lambda}_{\dot{a}}(q)\,(\sigma^\mu)^{\dot{a}a}\,\lambda_a(p)\,, \tag{H.25}$$

which is $\langle p^-|\,\gamma^\mu\,|q^-\rangle = \langle q^+|\,\gamma^\mu\,|p^+\rangle$, i.e. charge conjugation.

Hint. Use $\tilde{\lambda}^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{b}}$ and $\lambda^a = \epsilon^{ab}\lambda_b$.

Solution. We get

$$\begin{aligned}\lambda^a(p)\,(\bar{\sigma}^\mu)_{a\dot{a}}\,\tilde{\lambda}^{\dot{a}}(q) &= \epsilon^{ab}\lambda_b(p)\,(\bar{\sigma}^\mu)_{a\dot{a}}\,\epsilon^{\dot{a}\dot{b}}\tilde{\lambda}_{\dot{b}}(q)\\ &= \lambda_b(p)(-\epsilon^{\dot{b}\dot{a}})\,(\bar{\sigma}^\mu)^T_{\dot{a}a}\,\epsilon^{ab}\tilde{\lambda}_{\dot{b}}(q)\\ &= \tilde{\lambda}_{\dot{b}}(q)(\sigma^\mu)^{\dot{b}b}\lambda_b(p)\ .\end{aligned} \tag{H.26}$$

3. Prove the Fierz rearrangement

$$\left\langle k^-\right|\gamma^\mu\left|p^-\right\rangle\left\langle v^-\right|\gamma_\mu\left|q^-\right\rangle = 2\langle kv\rangle\,[qp]\ , \tag{H.27}$$

using charge conjugation $\langle v^-|\,\gamma^\mu\,|q^-\rangle = \langle q^+|\,\gamma^\mu\,|v^+\rangle$ and the Fierz identity for Pauli matrices $(\bar{\sigma}_\mu)_{a\dot{a}}(\sigma^\mu)^{\dot{b}b} = 2\delta^b_a\delta^{\dot{b}}_{\dot{a}}$.

Solution.

$$\begin{aligned}\left\langle k^-\right|\gamma^\mu\left|p^-\right\rangle\left\langle v^-\right|\gamma_\mu\left|q^-\right\rangle &= \left\langle k^-\right|\gamma^\mu\left|p^-\right\rangle\left\langle q^+\right|\gamma_\mu\left|v^+\right\rangle\\ &= \xi^\dagger_-(k)\bar{\sigma}^\mu\xi_-(p)\xi^\dagger_+(q)\sigma_\mu\xi_+(v)\\ &= \xi^{a*}_-(k)\bar{\sigma}^\mu_{a\dot{a}}\xi^{\dot{a}}_-(p)\xi^*_{+\dot{b}}(q)\sigma^{\dot{b}b}_\mu\xi_{+b}(v)\\ &= 2\xi^{*a}_-(k)\xi_{+a}(v)\xi^*_{+\dot{a}}(q)\xi^{\dot{a}}_-(p)\\ &= 2\langle kv\rangle[qp]\,.\end{aligned} \tag{H.28}$$

4. Prove the same Fierz rearrangement in terms of λ-spinors.

Solution. In complete analogy to exercise 3 above we have

$$\left\langle k^-\right|\gamma^\mu\left|p^-\right\rangle\left\langle q^+\right|\gamma_\mu\left|v^+\right\rangle = \lambda^a(k)\bar{\sigma}^\mu_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(p)\tilde{\lambda}_{\dot{b}}(q)\sigma^{\dot{b}b}_\mu\lambda_b(v)\,, \tag{H.29}$$

and all other computational steps are the same.

H.4 $e^-e^+ \to \mu^-\mu^+$ scattering

Consider the amplitude for $e^-e^+ \to \mu^-\mu^+$. We choose the momenta to be all outgoing, so that momentum conservation is $\sum_{i=1}^{4} p_i^\mu = 0$. Accordingly, helicites are for all outgoing momenta. i.e. an incoming left-handed electron is labelled as an outgoing right-handed positron, e.g. the amplitude, $e_L^-(-p_1)e_R^+(-p_2) \to \mu_R^-(p_3)\mu_L^+(p_4)$ becomes $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$ as shown in fig. H.1.

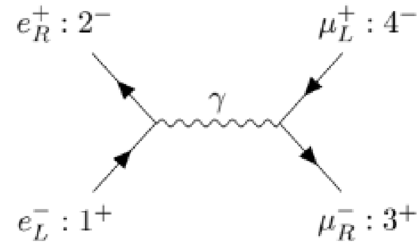


Figure H.1: $e^-e^+ \to \mu^-\mu^+$ scattering for the configuration $(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$.

1. Compute $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$.

   Solution. We have

$$\begin{aligned} M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) &= i\frac{e^2}{s_{12}} \left\langle 2^- \middle| \gamma_\mu \middle| 1^- \right\rangle \left\langle 3^+ \middle| \gamma^\mu \middle| 4^+ \right\rangle \\ \text{(use charge conj.)} \quad &= i\frac{e^2}{s_{12}} \left\langle 2^- \middle| \gamma_\mu \middle| 1^- \right\rangle \left\langle 4^- \middle| \gamma^\mu \middle| 3^- \right\rangle \\ \text{(use Fierz)} \quad &= i\frac{e^2}{s_{12}} 2\langle 24\rangle[31] \ . \end{aligned} \tag{H.30}$$

2. Compute $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$ using λ spinors.

   Solution. We have

$$\begin{aligned} M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) &= i\frac{e^2}{s_{12}} \lambda_2^a \bar\sigma^\mu_{a\dot a} \tilde\lambda_1^{\dot a} \tilde\lambda_{3\dot b} \sigma_\mu^{\dot b b} \lambda_{4b} \\ (\text{use } \bar\sigma^\mu_{a\dot a}\sigma_\mu^{\dot b b} = 2\delta_a{}^b \delta_{\dot a}{}^{\dot b}) \quad &= i\frac{e^2}{s_{12}} 2\lambda_2^a \tilde\lambda_1^{\dot a} \tilde\lambda_{3\dot a} \lambda_{4a} \\ &= i\frac{e^2}{s_{12}} 2\langle 24\rangle[31] \ . \end{aligned} \tag{H.31}$$

3. Write $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$ in terms of right-handed spinor products.

   Solution. Note that using $s = \langle 12\rangle[21]$, $s_{13} = s_{24}$ and momentum conservation $[21]\langle 13\rangle =$

$-[24]\langle 43\rangle$, we can rewrite

$$\begin{aligned}\frac{\langle 24\rangle[31]}{s_{12}} &= \frac{\langle 24\rangle[31]\langle 13\rangle}{\langle 12\rangle[21]\langle 13\rangle} \\ &= \frac{\langle 24\rangle[42]\langle 24\rangle}{\langle 12\rangle[21]\langle 13\rangle} \\ &= -\frac{\langle 24\rangle[42]\langle 24\rangle}{\langle 12\rangle[24]\langle 43\rangle} \\ &= -\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle}\,. \end{aligned} \tag{H.32}$$

4. Write $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$ in terms of left-handed spinor products.

   Solution. Here we use momentum conservation $[42]\langle 21\rangle = -[43]\langle 31\rangle$ and $s_{13} = s_{24}$ and get

$$\begin{aligned}\frac{\langle 24\rangle[31]}{s_{12}} &= \frac{\langle 24\rangle[31][42]}{[12]\langle 21\rangle[42]} \\ &= -\frac{[31]^2\langle 13\rangle}{[12][43]\langle 31\rangle} \\ &= -\frac{[13]^2}{[12][34]}. \end{aligned} \tag{H.33}$$

5. Do the same for $M_4(1^+_{e^+}, 2^-_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+})$. How is that amplitude related to $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+})$?

   Solution. In (H.30), we had

$$M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_{\mu^-}, 4^-_{\mu^+}) = i\frac{e^2}{s_{12}} \left\langle 2^-\right|\gamma_\mu\left|1^-\right\rangle \left\langle 4^-\right|\gamma^\mu\left|3^-\right\rangle \;.$$

   Our amplitude for the $M_4(1^+_{e^+}, 2^-_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+})$ reads

$$M_4(1^+_{e^+}, 2^-_{e^-}, 3^-_{\mu^-}, 4^+_{\mu^+}) = i\frac{e^2}{s_{12}} \left\langle 2^-\right|\gamma_\mu\left|1^-\right\rangle \left\langle 3^-\right|\gamma^\mu\left|4^-\right\rangle \;. \tag{H.34}$$

   As we can see, all the computation will be the same but we have to interchange $3 \leftrightarrow 4$. This is charge conjugation on the muon line.

H.5 Crossing symmetry

Use crossing symmetry to obtain the amplitudes for $eq \to eq$ scattering from the amplitudes for $e^-e^+ \to q\overline{q}$ scattering, as shown in fig. 1.3, and compute the amplitude squared and summed over the helicities.

Hint. Use the colour-stripped A_4 amplitudes for $e^-e^+ \to q\bar{q}$ scattering,

$$iM_4(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^-_{\bar{q}}) = i\frac{\langle 24\rangle\,[31]}{s_{12}} = i\frac{s_{13}}{s_{12}}e^{i\varphi}$$
$$iM_4(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_{\bar{q}}) = i\frac{\langle 23\rangle\,[41]}{s_{12}} = i\frac{s_{14}}{s_{12}}e^{i\varphi'}$$

Solution: Firstly, we remind that helicities are labelled for all outgoing momenta, e.g. the amplitude, $M_4(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^-_{\bar{q}})$, in the physical region becomes the amplitude for $e^-_L(-p_1)e^+_R(-p_2) \to q_R(p_3)\overline{q_L}(p_4)$. To get the helicity amplitude of $eq \to eq$ scattering from the amplitudes of $e^-e^+ \to \mu^-\mu^+$, or $e^-e^+ \to q\bar{q}$, one may use the crossing symmetry of the initial and final momenta, as shown in Fig. 1.3. We rewrite the Mandelstam invariants for the crossed diagram, following the correspondence,

$$\begin{aligned}(k_1+k_2)^2 = s \quad &\to \quad (\ell-\ell')^2 = t\,,\\ (k_1-k_3)^2 = t \quad &\to \quad (\ell-p')^2 = u\,,\\ (k_1-k_4)^2 = u \quad &\to \quad (\ell+p)^2 = s\,,\end{aligned} \tag{H.35}$$

Thus, we can obtain the helicity amplitudes of $eq \to eq$ by crossing the kinematic invariants of $e^-e^+ \to q\bar{q}$,

$$\begin{aligned}&\left\{\begin{matrix} e^-_R e^+_L \to \mu^-_R\mu^+_L, q_R\bar{q}_L \\ e^-_L e^+_R \to \mu^-_L\mu^+_R, q_L\bar{q}_R \end{matrix}\right\} \sim u^2 \quad \to \quad \left\{\begin{matrix} e^-_R q_R \to e^-_R q_R \\ e^-_L q_L \to e^-_L q_L \end{matrix}\right\} \sim s^2\\ &\left\{\begin{matrix} e^-_R e^+_L \to \mu^-_L\mu^+_R, q_L\bar{q}_R \\ e^-_L e^+_R \to \mu^-_R\mu^+_L, q_R\bar{q}_L \end{matrix}\right\} \sim t^2 \quad \to \quad \left\{\begin{matrix} e^-_R q_L \to e^-_R q_L \\ e^-_L q_R \to e^-_L q_R \end{matrix}\right\} \sim u^2,\end{aligned} \tag{H.36}$$

so the squared amplitude for $eq \to eq$ scattering becomes

$$\sum_{hel} |M|^2 = 8e^4 Q_q^2\ \frac{s^2+u^2}{t^2}\,. \tag{H.37}$$

The s^2 term comes from the scattering of L(R)-handed electrons on L(R)-handed quarks, and the u^2 term from the scattering of L(R)-handed electrons on R(L)-handed quarks.

H.6 Polarization vectors

The polarization vector of a photon or a gluon of momentum k and reference vector q is

$$\epsilon^{\pm *}_\mu(k,q) = \pm\frac{\langle q^\mp|\,\gamma_\mu\,|k^\mp\rangle}{\sqrt{2}\,\langle q^\mp|k^\pm\rangle}\,. \tag{H.38}$$

Show that it fulfils the usual properties of polarization vectors:

1. Show $\epsilon_\mu^{\pm *}(k,q) = \epsilon_\mu^{\mp}(k,q)$.

   Solution. We have:

$$
\begin{aligned}
\left(\epsilon_\mu^{+*}(k,q)\right)^* &= \left(\frac{\langle q^-|\,\gamma_\mu\,|k^-\rangle}{\sqrt{2}\langle qk\rangle}\right)^* \\
&= \frac{\langle k^-|\,\gamma_\mu\,|q^-\rangle}{\sqrt{2}[kq]} \\
&= -\frac{\langle q^+|\,\gamma_\mu\,|k^+\rangle}{\sqrt{2}[qk]} \\
&= \epsilon_\mu^{-*}(k,q)\,.
\end{aligned}
\tag{H.39}
$$

2. Show $(k\cdot\epsilon^{\pm}(k,q)) = 0$.

   Solution. We have:

$$
\begin{aligned}
\left(k\cdot\epsilon^-(k,q)\right) &= \left(k\cdot\epsilon^{+*}(k,q)\right) \\
&= \frac{\langle q|\,\gamma_\mu|k]}{\sqrt{2}\langle qk\rangle}k^\mu \\
\text{(use: } \not{p} = |p\rangle\,[p|+|p]\,\langle p|) \qquad &= \frac{\langle qk\rangle[kk]}{\sqrt{2}\langle qk\rangle} \\
&= 0\,.
\end{aligned}
\tag{H.40}
$$

   ϵ^+ works in complete analogy.

3. Show $\epsilon^h(k,q)\cdot\epsilon^{h'*}(k,q) = -\delta^{hh'}$. Why does that imply $\epsilon^h(k,q)\cdot\epsilon^h(k,q) = 0$?

   Solution. We have:

$$
\begin{aligned}
\epsilon^{+*}(k,q)\epsilon^{+}(k,q) &= \epsilon^{+*}(k,q)\epsilon^{-*}(k,q) \\
&= -\frac{1}{2\langle qk\rangle[qk]}\left\langle q^-\right|\gamma_\mu\left|k^-\right\rangle\left\langle q^+\right|\gamma^\mu\left|k^+\right\rangle \\
&= -\frac{1}{2\langle qk\rangle[qk]}\left\langle q^-\right|\gamma_\mu\left|k^-\right\rangle\left\langle k^-\right|\gamma^\mu\left|q^-\right\rangle \\
&= -\frac{1}{2\langle qk\rangle[qk]}2\langle qk\rangle[qk] \\
&= -1\,.
\end{aligned}
\tag{H.41}
$$

and

$$
\begin{aligned}
\epsilon^{+*}(k,q)\epsilon^{-}(k,q) &= \epsilon^{+*}(k,q)\epsilon^{+*}(k,q) \\
&= \left(\frac{1}{\sqrt{2}\langle qk\rangle}\left\langle q^{-}\right|\gamma_\mu\left|k^{-}\right\rangle\right)^2 \\
&= 0\,, \qquad \text{(H.42)}
\end{aligned}
$$

since the current is nilpotent.

4. Show that the usual polarization sum is fulfilled: $\sum_h \epsilon_h^\mu(p,k)\cdot\epsilon_h^{\nu*}(p,k) = -g^{\mu\nu} + \frac{p^\mu k^\nu + p^\nu k^\mu}{(p\cdot k)}$.
Solution. We have

$$
\begin{aligned}
\epsilon_\nu^{+*}(p,k) &= \frac{\tilde\lambda_{\dot a}(p)\sigma_\nu^{\dot a a}\lambda_a(k)}{\sqrt{2}\langle kp\rangle}\,, \\
\epsilon_\mu^{-*}(p,k) &= -\frac{\tilde\lambda_{\dot b}(k)\sigma_\mu^{\dot b b}\lambda_b(p)}{\sqrt{2}[kp]}\,. \qquad \text{(H.43)}
\end{aligned}
$$

and our polarization sum may be written as

$$
\begin{aligned}
\epsilon_\mu^{+}\epsilon_\nu^{+*} + \epsilon_\mu^{-}\epsilon_\nu^{-*} &= \epsilon_\mu^{-*}\epsilon_\nu^{+*} + (\epsilon_\mu^{-*}\epsilon_\nu^{+*})^* \\
&= 2\,\mathrm{Re}(\epsilon_\mu^{-*}\epsilon_\nu^{+*})\,. \qquad \text{(H.44)}
\end{aligned}
$$

However, we see that $\epsilon_\mu^{-*}\epsilon_\nu^{+*}$ contains the two spinors,

$$
\begin{aligned}
\lambda_b(p)\tilde\lambda_{\dot a}(p) &= (p\bar\sigma)_{b\dot a}\,, \\
\lambda_a(k)\tilde\lambda_{\dot b}(k) &= (k\bar\sigma)_{a\dot b}\,. \qquad \text{(H.45)}
\end{aligned}
$$

From that it follows

$$
\begin{aligned}
\epsilon_\mu^{-*}(p,k)\epsilon_\nu^{+*}(p,k) &\propto (p\bar\sigma)_{b\dot a}\sigma_\nu^{\dot a a}(k\bar\sigma)_{a\dot b}\sigma_\mu^{\dot b b} \\
&= p^\alpha k^\beta\,\mathrm{tr}(\sigma_\mu\bar\sigma_\alpha\sigma_\nu\bar\sigma_\beta)\,. \qquad \text{(H.46)}
\end{aligned}
$$

where

$$
\mathrm{tr}(\sigma_\mu\bar\sigma_\alpha\sigma_\nu\bar\sigma_\beta) = 2\left(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta} - i\epsilon_{\mu\alpha\nu\beta}\right)\,. \qquad \text{(H.47)}
$$

So

$$
\epsilon_\mu^{-*}(p,k)\epsilon_\nu^{+*}(p,k) \propto 2\left(p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu}(kp) - i\epsilon_{\mu\alpha\nu\beta}p^\alpha k^\beta\right)\,. \qquad \text{(H.48)}
$$

The ϵ-tensor drops out due to (H.44), and we see from the prefactor, that we will reproduce the known polarization sum.

H.7 Polarization vectors in 2x2 matrix notation

Compute $\epsilon^{-*}(p,k)_{a\dot{a}} = (\epsilon^{-*}(p,k)\cdot\bar{\sigma})_{a\dot{a}}$.

Solution. We have:

$$\begin{aligned}\epsilon_\mu^{-*}(p,k) &= -\frac{\langle k^+|\,\gamma_\mu\,|p^+\rangle}{\sqrt{2}\,\langle k^+|p^-\rangle}\\ &= -\frac{[k|\gamma_\mu\,|p\rangle}{\sqrt{2}[kp]}\\ &= -\frac{\tilde{\lambda}_{\dot{a}}(k)\sigma_\mu^{\dot{a}a}\lambda_a(p)}{\sqrt{2}[kp]}\,. \end{aligned} \tag{H.49}$$

This means

$$\begin{aligned}\epsilon_{b\dot{b}}^{-*}(p,k) &= -\frac{\tilde{\lambda}_{\dot{a}}(k)\sigma_\mu^{\dot{a}a}\lambda_a(p)}{\sqrt{2}[kp]}(\bar{\sigma}^\mu)_{b\dot{b}}\\ \text{(use: } (\bar{\sigma}_\mu)_{a\dot{a}}(\sigma^\mu)^{\dot{b}b} = 2\delta_a{}^b\delta_{\dot{a}}{}^{\dot{b}}) \quad &= -\sqrt{2}\frac{\tilde{\lambda}_{\dot{b}}(k)\lambda_b(p)}{[kp]}\,. \end{aligned} \tag{H.50}$$

Furthermore:

$$\begin{aligned}\epsilon_\mu^{+*}(p,k) &= \frac{\langle k^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\,\langle k^-|p^+\rangle}\\ &= \frac{\langle p^+|\,\gamma_\mu\,|k^+\rangle}{\sqrt{2}\langle kp\rangle}\\ &= \frac{\tilde{\lambda}_{\dot{a}}(p)\sigma_\mu^{\dot{a}a}\lambda_a(k)}{\sqrt{2}\langle kp\rangle}\,, \end{aligned} \tag{H.51}$$

which means

$$\epsilon_{b\dot{b}}^{+*}(p,k) = \sqrt{2}\frac{\tilde{\lambda}_{\dot{b}}(p)\lambda_b(k)}{\langle kp\rangle}\,. \tag{H.52}$$

H.8 Polarization vector identities

Show that with the representation (H.38) of the polarisation vector, the following identities hold:

1.
$$q\cdot\epsilon^{\pm}(k,q) = 0\,. \tag{H.53}$$

Solution. We have:

$$
\begin{aligned}
q\cdot\epsilon^-(k,q) &= q\cdot\epsilon^{+*}(k,q)\\
&= \frac{\langle q|\,\gamma_\mu|k]}{\sqrt{2}\langle qk\rangle}q^\mu\\
\text{(use: } \not{q} = |q\rangle\,[q|+|q]\,\langle q|) \qquad &= \frac{\langle qq\rangle[qk]}{\sqrt{2}\langle qk\rangle}\\
&= 0\,.
\end{aligned}
\tag{H.54}
$$

2.

$$
\epsilon^{\pm}(p_i,k)\cdot\epsilon^{\pm}(p_j,k) = 0\,, \tag{H.55}
$$

$$
\epsilon^{+*}(p_i,p_j)\cdot\epsilon^{-*}(p_j,k) = 0\,, \tag{H.56}
$$

$$
\epsilon^{+*}(p_i,k)\cdot\epsilon^{-*}(p_j,p_i) = 0\,. \tag{H.57}
$$

Solution. We have:

$$
\begin{aligned}
\epsilon^{+*}(p_i,k)\cdot\epsilon^{+*}(p_j,k) &\propto \left\langle k^-\right|\gamma_\mu\left|p_i^-\right\rangle\left\langle k^-\right|\gamma^\mu\left|p_j^-\right\rangle\\
&= \langle kk\rangle[p_jp_i]\\
&= 0
\end{aligned}
\tag{H.58}
$$

$$
\begin{aligned}
\epsilon^{+*}(p_i,p_j)\cdot\epsilon^{-*}(p_j,k) &\propto \left\langle p_j^-\right|\gamma_\mu\left|p_i^-\right\rangle\left\langle k^+\right|\gamma^\mu\left|p_j^+\right\rangle\\
&= \left\langle p_j^-\right|\gamma_\mu\left|p_i^-\right\rangle\left\langle p_j^-\right|\gamma^\mu\left|k^-\right\rangle\\
&= \langle p_jp_j\rangle[kp_i]\\
&= 0\,,
\end{aligned}
\tag{H.59}
$$

$$
\begin{aligned}
\epsilon^{+*}(p_i,k)\cdot\epsilon^{-*}(p_j,p_i) &\propto \left\langle k^-\right|\gamma_\mu\left|p_i^-\right\rangle\left\langle p_i^+\right|\gamma^\mu\left|p_j^+\right\rangle\\
&\propto \left\langle k^-\right|\gamma_\mu\left|p_i^-\right\rangle\left\langle p_j^-\right|\gamma^\mu\left|p_i^-\right\rangle\\
&= \langle kp_j][p_ip_i]\\
&= 0\,.
\end{aligned}
\tag{H.60}
$$

3. $(\gamma\cdot\epsilon^{+*}(p,k)) = \dfrac{\sqrt{2}}{\langle kp\rangle}\left(\left|p^-\right\rangle\!\left\langle k^-\right| + \left|k^+\right\rangle\!\left\langle p^+\right|\right)$.

Solution. We have

$$
\begin{aligned}
\gamma^\mu\epsilon^{+*}_\mu(p,k) &= \frac{\langle k^-|\,\gamma_\mu\,|p^-\rangle}{\sqrt{2}\,\langle k^-|p^+\rangle}\gamma^\mu\\
&= \frac{1}{\sqrt{2}\langle kp\rangle}\lambda^a(k)(\bar\sigma_\mu)_{a\dot a}\tilde\lambda^{\dot a}(p)\begin{pmatrix} 0 & (\sigma^\mu)^{\dot b b}\\ (\bar\sigma^\mu)_{b\dot b} & 0\end{pmatrix}\,.
\end{aligned}
\tag{H.61}
$$

For the upper block we use the Fierz identity $(\bar{\sigma}_\mu)_{a\dot{a}}(\sigma^\mu)^{\dot{b}b} = 2\delta_a{}^b\delta_{\dot{a}}{}^{\dot{b}}$ and get

$$\frac{\sqrt{2}}{\langle kp\rangle}\tilde{\lambda}^{\dot{b}}(p)\lambda(k)^b = \frac{\sqrt{2}}{\langle kp\rangle}\left|p^-\right\rangle\left\langle k^-\right| . \tag{H.62}$$

For the lower block we use:

$$\begin{aligned}(\bar{\sigma}_\mu)_{a\dot{a}}(\bar{\sigma}_\mu)_{b\dot{b}} &= (\bar{\sigma}_\mu)_{a\dot{a}}(-\epsilon_{bd})\epsilon_{\dot{c}\dot{b}}(\sigma^\mu)^{\dot{c}d} \\ &= 2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} , \\ \Rightarrow \frac{\sqrt{2}}{\langle kp\rangle}\lambda_b(k)\tilde{\lambda}_{\dot{b}}(p) &= \frac{\sqrt{2}}{\langle kp\rangle}\left|k^+\right\rangle\left\langle p^+\right| ,\end{aligned} \tag{H.63}$$

which then yields

$$\left(\gamma\cdot\epsilon^{+*}(p,k)\right) = \frac{\sqrt{2}}{\langle kp\rangle}\left(\left|p^-\right\rangle\!\left\langle k^-\right| + \left|k^+\right\rangle\!\left\langle p^+\right|\right) . \tag{H.64}$$

We can furthermore look at

$$\begin{aligned}\epsilon_\mu^{-*}(p,k) &= -\frac{\langle k^+|\,\gamma_\mu\,|p^+\rangle}{\sqrt{2}\,\langle k^+|p^-\rangle} \\ &= -\frac{\langle p^-|\,\gamma_\mu\,|k^-\rangle}{\sqrt{2}[kp]} \\ &= -\frac{1}{\sqrt{2}[kp]}\lambda^a(p)(\bar{\sigma}_\mu)_{a\dot{a}}\tilde{\lambda}^{\dot{a}}(k) ,\end{aligned} \tag{H.65}$$

which then gives with the analogous computation

$$\epsilon_\mu^{-*}(p,k)\gamma^\mu = -\frac{\sqrt{2}}{[kp]}\left(\left|k^-\right\rangle\!\left\langle p^-\right| + \left|p^+\right\rangle\!\left\langle k^+\right|\right) . \tag{H.66}$$

4.

$$\not{\epsilon}^{\pm*}(p_i,p_j)\left|p_j^\pm\right\rangle = \left\langle p_j^\mp\right|\not{\epsilon}^\pm(p_i,p_j) = 0 . \tag{H.67}$$

Solution. Use eq. (H.64), noticing that $\langle kk\rangle = [kk] = 0$.

H.9 Charge conjugation of currents

Show that

$$\left\langle k^\pm\right|\gamma_\mu\gamma_\nu\left|q^\mp\right\rangle = -\left\langle q^\pm\right|\gamma_\nu\gamma_\mu\left|k^\mp\right\rangle . \tag{H.68}$$

Hint. Use $\xi_-^* = -i\sigma^2\xi_+$ and $\sigma^2\bar{\sigma}^\mu\sigma^2 = \sigma^{\mu T}$.

Solution. We have:

$$\begin{aligned}
\left\langle k^- \middle| \gamma_\mu \gamma_\nu \middle| q^+ \right\rangle &= \lambda^a(k) (\bar\sigma_\mu)_{a\dot a} (\sigma_\nu)^{\dot a b} \lambda_b(q) \\
&= \lambda^a(k) \left((\sigma_\mu)^{\dot c c} \epsilon_{ac} \epsilon_{\dot a \dot c} \right) \left((\bar\sigma_\nu)_{d\dot d} \epsilon^{bd} \epsilon^{\dot a \dot d} \right) \lambda_b(q) \\
&= (\sigma_\mu)^{\dot c c} (\bar\sigma_\nu)_{d\dot d} \left(- \underbrace{\epsilon^{\dot d \dot a} \epsilon_{\dot a \dot c}}_{-\delta^{\dot d}_{\dot c}} \right) \left(-\epsilon_{ca} \lambda^a(k) \right) \left(\lambda_b(q) \epsilon^{bd} \right) \\
&= - \left\langle q^- \middle| \gamma_\nu \gamma_\mu \middle| k^+ \right\rangle . \qquad \text{(H.69)}
\end{aligned}$$

In particular notice that for any even number of gamma matrices $\left(-\epsilon_{ca} \lambda^a(k) \right) \left(\lambda_b(q) \epsilon^{bd} \right)$ will always appear and we will have

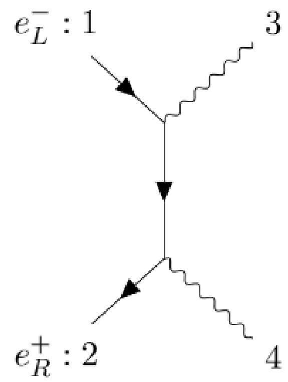
$$\left\langle k^\pm \middle| \gamma_\mu \dots \gamma_\nu \middle| q^\mp \right\rangle = - \left\langle q^\pm \middle| \gamma_\nu \dots \gamma_\mu \middle| k^\mp \right\rangle . \qquad \text{(H.70)}$$

Similar considerations show for an odd number of gamma matrices

$$\left\langle k^\pm \middle| \gamma_\mu \dots \gamma_\nu \middle| q^\pm \right\rangle = \left\langle q^\mp \middle| \gamma_\nu \dots \gamma_\mu \middle| k^\mp \right\rangle . \qquad \text{(H.71)}$$

H.10 $e^- e^+ \to \gamma\gamma$

Consider the process $e_L^- e_R^+ \to \gamma\gamma$ with the amplitude $M_4(1^+_{e^+}, 2^-_{e^-}, 3_\gamma, 4_\gamma)$ and a sample diagram shown below. The other relevant diagram is obtained by crossing of the photon legs.



If we only fix the electron and positron helicities, there are still four helicity configurations for the photons: 3^+4^+, 3^-4^-, 3^+4^- and 3^-4^+. Which of them are related by parity transformations and swapping of labels? Compute the independent ones.

Hint. For 3^+4^+, use p_2 as a reference vector. For 3^+4^-, use $\epsilon^+(3,2)$ and $\epsilon^-(4,1)$.

Solution. The 3^+4^+ and 3^-4^- configurations are related by parity, while 3^+4^- and 3^-4^+ by swapping of the labels. Therefore we have to compute 3^+4^+ and 3^+4^- only.

For the sake of brevity, in the following we denote ϵ^* as ϵ.

We have

$$iM_4(1^+_{e^+}, 2^-_{e^-}, 3_\gamma, 4_\gamma) = (-ie)^2 \langle 2^- \big| \left(\not\epsilon(p_4) \frac{i(\not p_2 + \not p_4)}{s_{24}} \not\epsilon(p_3) + \not\epsilon(p_3) \frac{i(\not p_2 + \not p_3)}{s_{23}} \not\epsilon(p_4) \right) \big| 1^- \rangle \,. \tag{H.72}$$

For the 3^+4^+ configuration we see that by choosing the reference momentum for both polarization as p_2, we will have $\langle 2^- | \not\epsilon(k, p_2) = 0$. So the complete amplitude contribution vanishes.

For the 3^+4^- configuration we see that by choosing the polarization vectors $\epsilon^+(3,2)$ and $\epsilon^-(4,1)$, we can achieve the second term to vanish again, since $\langle 2^- | \not\epsilon(3,2) = 0$. The amplitude then reads:

$$\begin{aligned} iM_4(1^+_{e^+}, 2^-_{e^-}, 3_\gamma, 4_\gamma) &= (-ie)^2 \langle 2^- \big| \left(\not\epsilon^-(4,1) \frac{i(\not p_2 + \not p_4)}{s_{24}} \not\epsilon^+(3,2) \right) \big| 1^- \rangle \\ &= \frac{-ie^2}{s_{24}} \langle 2^- \big| \left(-\frac{\sqrt{2}}{[14]} \big|4^+\rangle \langle 1^+\big| \right) \left(\big|2^-\rangle\langle 2^-\big| + \big|4^-\rangle\langle 4^-\big| \right) \left(\frac{\sqrt{2}}{\langle 23\rangle} \big|2^+\rangle\langle 3^+\big| \right) \big|1^-\rangle \\ &= \frac{ie^2}{s_{13}} \frac{2}{[14]\langle 23\rangle} \langle 24\rangle[14]\langle 42\rangle[31] \\ &= -2ie^2 \frac{\langle 24\rangle^2}{\langle 13\rangle\langle 23\rangle} \,. \end{aligned} \tag{H.73}$$

H.11 $e^-_L e^+_R \to q_R g_R \bar{q}_L$ Scattering

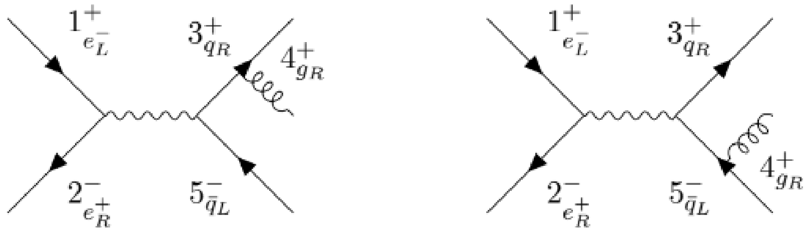


Figure H.2: $e^-(-p_1)e^+(-p_2) \to q(p_3)g(p_4)\bar{q}(p_5)$

In the following we want to consider the scattering amplitude for the process $e^-(-p_1)e^+(-p_2) \to q(p_3)g(p_4)\bar{q}(p_5)$ with the contributing diagrams shown in fig. H.2. We may write the amplitude as

$$M_5(e^-e^+ \to qg\bar{q}) = (\sqrt{2}e)^2 Q_q Q_e (T^a_4)^{i_5}_{i_3} A_5\left(e^-e^+ \to qg\bar{q}\right) \,. \tag{H.74}$$

where A_5 denotes the color-stripped amplitude. In the following use the normalization of the fundamental representation of $SU(3)$ such that $\mathrm{Tr}\big(T^aT^b\big) = T_F\delta^{ab}$ with $T_F = 1$, which results in the quark-gluon-vertex $i\frac{g}{\sqrt{2}}T^a\gamma^\nu$.

1. Compute the color-stripped amplitude $A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}})$

Hint Choose the reference vector of the polarization $q = p_5$. The product

$$\gamma^\alpha \gamma^\beta \gamma^\delta = \begin{pmatrix} 0 & \sigma^\alpha \bar{\sigma}^\beta \sigma^\delta \\ \bar{\sigma}^\alpha \sigma^\beta \bar{\sigma}^\delta & 0 \end{pmatrix} \tag{H.75}$$

can be used to rewrite the current in 2-dimensional notation.

Solution. We denote the outgoing gluon polarization ϵ^* by ϵ and have

$$\begin{aligned} &iA_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) \\ &= -\frac{i}{2\sqrt{2}s_{12}} \left\langle 2^- \right| \gamma^\mu \left| 1^- \right\rangle \left[\frac{\langle 3^+ | \, \gamma_\nu \left(\not{p}_3 + \not{p}_4 \right) \gamma_\mu \, |5^+\rangle}{s_{34}} + \frac{\langle 3^+ | \, \gamma_\mu \left(\not{p}_4 + \not{p}_5 \right) \gamma_\nu \, |5^+\rangle}{s_{45}} \right] \epsilon^{+\nu} \left(p_4, q \right). \end{aligned} \tag{H.76}$$

We can now use

$$\gamma \cdot \epsilon^+(p,k) = \frac{\sqrt{2}}{\langle kp \rangle} \left(\left| p^- \right\rangle\!\left\langle k^- \right| + \left| k^+ \right\rangle\!\left\langle p^+ \right| \right) , \tag{H.77}$$

and see that choosing our reference vector as $q = p_5$ will make the second term vanish. We therefore have

$$iA_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) = -\frac{i}{2s_{12}} \left\langle 2^- \right| \gamma^\mu \left| 1^- \right\rangle \frac{[34]}{\langle 54 \rangle} \frac{\langle 5^- | \left(\not{p}_3 + \not{p}_4 \right) \gamma_\mu \, |5^+\rangle}{s_{34}}, \tag{H.78}$$

with

$$\begin{aligned} \left\langle 2^- \right| \gamma_\mu \left| 1^- \right\rangle \left\langle 5^- \right| \gamma_\rho \gamma^\mu \left| 5^+ \right\rangle &= \lambda_2^a \left(\bar{\sigma}_\mu \right)_{a\dot{a}} \tilde{\lambda}_1^{\dot{a}} \lambda_5^b \left(\bar{\sigma}_\rho \right)_{b\dot{b}} \left(\sigma^\mu \right)^{\dot{b}c} \lambda_{5c} \\ \left(\text{use: } \left(\bar{\sigma}_\mu \right)_{a\dot{a}} \left(\sigma^\mu \right)^{\dot{b}c} = 2\delta_a^c \delta_{\dot{a}}^{\dot{b}} \right) \qquad &= 2\lambda_2^a \lambda_{5a} \lambda_5^b \left(\bar{\sigma}_\rho \right)_{b\dot{b}} \tilde{\lambda}_1^{\dot{b}} \\ &= 2\langle 25 \rangle \left\langle 5^- \right| \gamma_\rho \left| 1^- \right\rangle , \end{aligned} \tag{H.79}$$

and momentum conservation $\langle 5^- | \left(\not{p}_3 + \not{p}_4 \right) |1^-\rangle = - \langle 5^- | \left(\not{p}_1 + \not{p}_2 + \not{p}_5 \right) |1^-\rangle = - \langle 5^- | \not{p}_2 \, |1^-\rangle$ we obtain

$$\begin{aligned} iA_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) &= \frac{i}{s_{12}s_{34}} \frac{[34]\langle 25 \rangle}{\langle 54 \rangle} \langle 52 \rangle [21] \\ &= i \frac{[43]\langle 25 \rangle^2 [21]}{\langle 12 \rangle [21] \langle 34 \rangle [43] \langle 54 \rangle} \\ &= -i \frac{\langle 25 \rangle^2}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle} . \end{aligned} \tag{H.80}$$

2. Compute the color-stripped amplitude $A_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_g, 5^+_{\bar{q}})$ from scratch and verify that your result is the same as $A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}})$ with charge conjugation on the quark line.

Solution. We denote the outgoing gluon polarization ϵ^* by ϵ and have

$$iA_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_g, 5^+_{\bar{q}}) \\ = -\frac{i}{2\sqrt{2}s_{12}} \left\langle 2^- \middle| \gamma^\mu \middle| 1^- \right\rangle \left[\frac{\langle 3^- | \gamma_\nu \left(\not{p}_3 + \not{p}_4 \right) \gamma_\mu | 5^- \rangle}{s_{34}} + \frac{\langle 3^- | \gamma_\mu \left(\not{p}_4 + \not{p}_5 \right) \gamma_\nu | 5^- \rangle}{s_{45}} \right] \epsilon^{+\nu} (p_4, q) . \tag{H.81}$$

For a odd number of γ-matrices we have the charge conjugation

$$\left\langle k^\pm \middle| \gamma_\mu \ldots \gamma_\nu \middle| q^\pm \right\rangle = \left\langle q^\mp \middle| \gamma_\nu \ldots \gamma_\mu \middle| k^\mp \right\rangle , \tag{H.82}$$

which we can apply to the quark currents in the squared brackets to obtain

$$iA_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_g, 5^+_{\bar{q}}) \\ = -\frac{i}{2\sqrt{2}s_{12}} \left\langle 2^- \middle| \gamma^\mu \middle| 1^- \right\rangle \left[\frac{\langle 5^+ | \gamma_\nu \left(\not{p}_4 + \not{p}_5 \right) \gamma_\mu | 3^+ \rangle}{s_{45}} + \frac{\langle 5^+ | \gamma_\mu \left(\not{p}_3 + \not{p}_4 \right) \gamma_\nu | 3^+ \rangle}{s_{34}} \right] \epsilon^{+\nu} (p_4, q) . \tag{H.83}$$

If we compare (H.83) with (H.76), we see that we indeed have

$$iA_5(1^+_{e^+}, 2^-_{e^-}, 3^-_q, 4^+_g, 5^+_{\bar{q}}) = iA_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}})|_{3\leftrightarrow 5} \\ = -i \frac{\langle 23 \rangle^2}{\langle 12 \rangle \langle 34 \rangle \langle 45 \rangle} . \tag{H.84}$$

3. Compute the colour-stripped amplitude $A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^-_g, 5^-_{\bar{q}})$ from scratch and verify that your result is same as in the lecture, where it was obtained from $A_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}})$ with charge conjugation and parity transformation.

Solution. We denote the outgoing gluon polarization ϵ^* by ϵ and have

$$iA_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) \\ = -\frac{i}{2\sqrt{2}s_{12}} \left\langle 2^- \middle| \gamma^\mu \middle| 1^- \right\rangle \left[\frac{\langle 3^+ | \gamma_\nu \left(\not{p}_3 + \not{p}_4 \right) \gamma_\mu | 5^+ \rangle}{s_{34}} + \frac{\langle 3^+ | \gamma_\mu \left(\not{p}_4 + \not{p}_5 \right) \gamma_\nu | 5^+ \rangle}{s_{45}} \right] \epsilon^{-\nu} (p_4, q) . \tag{H.85}$$

We now use

$$\epsilon^{-*}_\mu(p, k)\gamma^\mu = -\frac{\sqrt{2}}{[kp]} \left(\left| k^- \right\rangle\left\langle p^- \right| + \left| p^+ \right\rangle\left\langle k^+ \right| \right) , \tag{H.86}$$

and see that the choice of the reference vector $q = p_3$ lets the first term vanish, and the amplitude becomes

$$iA_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) = \frac{i\langle 45\rangle}{2s_{12}[34]} \left\langle 2^- \right| \gamma^\mu \left| 1^- \right\rangle \frac{\langle 3^+ | \gamma_\mu \left(\not{p}_4 + \not{p}_5 \right) | 3^- \rangle}{s_{45}} . \tag{H.87}$$

Using again the Fierz identity for Pauli matrices as above

$$\left\langle 2^- \right| \gamma^\mu \left| 1^- \right\rangle \left\langle 3^+ \right| \gamma_\mu \gamma_\nu \left| 3^- \right\rangle = 2[31] \left\langle 2^- \right| \gamma_\nu \left| 3^- \right\rangle . \tag{H.88}$$

we find

$$\begin{aligned} &iA_5(1^+_{e^+}, 2^-_{e^-}, 3^+_q, 4^+_g, 5^-_{\bar{q}}) \\ &= \frac{i\langle 45\rangle[31]}{s_{12}[34]} \left\langle 2^- \right| \gamma^\mu \left| 1^- \right\rangle \frac{\langle 2^- | \left(\not{p}_4 + \not{p}_5 \right) | 3^- \rangle}{s_{45}} \\ &= -\frac{i\langle 45\rangle[31]}{s_{12}[34]} \frac{\langle 2^- | \left(\not{p}_1 + \not{p}_2 + \not{p}_3 \right) | 3^- \rangle}{s_{45}} \\ &= -\frac{i\langle 45\rangle[31]}{s_{12}[34]} \frac{\langle 21\rangle[13]}{s_{45}} \\ &= -i\frac{[13]^2}{[12][34][45]} . \end{aligned} \tag{H.89}$$

H.12 $q\bar{q} \to gg$ scattering

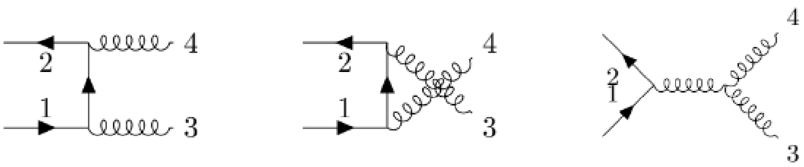


Figure H.3: $q_L(-p_1)\bar{q}_R(-p_2) \to g(p_3)g(p_4)$ scattering.

The amplitude for $q_L(-p_1)\bar{q}_R(-p_2) \to g(p_3)g(p_4)$ can be written as

$$M(1^+_{\bar{q}}, 2^-_q, 3_g, 4_g) = g^2 \left[A(1^+_{\bar{q}}, 2^-_q, 3_g, 4_g) T^{a_3} T^{a_4} + A(1^+_{\bar{q}}, 2^-_q, 4_g, 3_g) T^{a_4} T^{a_3} \right] . \tag{H.90}$$

with the contributing diagrams shown in fig. H.3.

Compute the colour-ordered amplitude $A(1^+_{\bar{q}}, 2^-_q, 3_g, 4_g)$.

We have four helicity configurations for the gluons: 3^+4^+, 3^-4^-, 3^+4^- and 3^-4^+. The first two are related by parity. The other two correspond to different colour-ordered amplitudes. Compute the 3^+4^+, 3^+4^- and 3^-4^+ configurations.

Hint: Use the amplitude of $e^-e^+ \to \gamma\gamma$ for the Abelian part.

Hint: Use the colour-ordered Feynman rule $3g$-vertex $3g \to \frac{ig}{\sqrt{2}} \operatorname{tr}\left(T^a T^b T^c\right)(g^{\mu\rho}(q-k)^\nu +$

$g^{\mu\nu}(k-p)^\rho + g^{\rho\nu}(p-q)^\mu)$ with (g_1, g_2, g_3) carrying the outgoing momentum (k,p,q), Lorentz indexes (μ,ν,ρ) and colour-indices (a,b,c).

Solution. We have the amplitude of $e^-e^+ \to \gamma\gamma$

$$iM_4(1^+_{e^+}, 2^-_{e^-}, 3_\gamma, 4_\gamma) = (-ie)^2 \langle 2^- \big| \left(\not\epsilon(p_4)\frac{i(\not p_2 + \not p_4)}{s_{24}}\not\epsilon(p_3) + \not\epsilon(p_3)\frac{i(\not p_2 + \not p_3)}{s_{23}}\not\epsilon(p_4)\right) \big|1^-\rangle \ . \tag{H.91}$$

from which we obtain the Abelian part of $q\bar{q} \to gg$

$$iM_4(1^+_{\bar{q}}, 2^-_q, 3_g, 4_g) = (i\frac{g}{\sqrt{2}})^2 \langle 2^- \big| \left(\not\epsilon(p_4)\frac{i(\not p_2 + \not p_4)}{s_{24}}\not\epsilon(p_3)T^{a_4}T^{a_3} + \not\epsilon(p_3)\frac{i(\not p_2 + \not p_3)}{s_{23}}\not\epsilon(p_4)T^{a_3}T^{a_4}\right) \big|1^-\rangle \ . \tag{H.92}$$

We select the $T^{a_3}T^{a_4}$ piece, only the second term contributes to it. Then we add the non-Abelian piece

$$\begin{aligned} & iM_4(1^+_{\bar{q}}, 2^-_q, 3_g, 4_g)|_{T^3T^4} \\ = \ & (i\frac{g}{\sqrt{2}})^2 \langle 2^- \big| \Big(\not\epsilon(p_3)\frac{i(\not p_2 + \not p_3)}{s_{23}}\not\epsilon(p_4)T^3T^4 + \frac{-i}{s_{12}}T^c \operatorname{tr}\big(T^3T^4T^c\big) \\ & \cdot \big[\not\epsilon_4((p_4 - (p_1+p_2))\cdot\epsilon_3) + \not\epsilon_3((-p_3 + (p_1+p_2))\cdot\epsilon_4) + (\not p_3 + \not p_4)(\epsilon_3\cdot\epsilon_4)\big]\Big) \big|1^-\rangle \ , \end{aligned} \tag{H.93}$$

where we use the shorthand notation $\epsilon(p_i) = \epsilon_i$. We use momentum conservation, the Fierz identity on the T^a matrices and strip off the coupling constant. The colour ordered amplitude is

$$\begin{aligned} & iA_4(1^+_{\bar{q}}, 2^-_q, 3_g, 4_g) \\ = \ & \left(\frac{i}{\sqrt{2}}\right)^2 \langle 2^- \big| \left(\not\epsilon(p_3)\frac{i(\not p_2 + \not p_3)}{s_{23}}\not\epsilon(p_4) + \frac{-i}{s_{12}}\left[2\not\epsilon_4(p_4\cdot\epsilon_3) - 2\not\epsilon_3(p_3\cdot\epsilon_4) + 2\not p_3(\epsilon_3\cdot\epsilon_4)\right]\right) \big|1^-\rangle \ . \end{aligned} \tag{H.94}$$

1. For 3^+4^+, choose $\epsilon^+_\mu(i,2)$ with $i = 3,4$.

   Solution. If the reference vector is chosen to be $q = p_2$ we have by eqs. (H.55) and (H.67) that $(\epsilon^+_3(2)\epsilon^+_4(2)) = 0$ and $\langle 2^- | \not\epsilon^+_i(2) = 0$ and the amplitude vanishes.

2. For 3^+4^-, choose $\epsilon^+_\mu(3,2)$ and $\epsilon^-_\mu(4,3)$. Repeat the computation with $\epsilon^+(3,4)$. Although the result is the same by gauge invariance, see how different terms determine it.

   Solution. For 3^+4^- we choose

$$\epsilon^+_{3,\mu}(2) = \frac{\langle 2^- | \gamma_\mu | 3^- \rangle}{\sqrt{2}\langle 23\rangle} \ , \tag{H.95}$$

$$\epsilon^-_{4,\mu}(3) = -\frac{\langle 3^+ | \gamma_\mu | 4^+ \rangle}{\sqrt{2}[34]} \ , \tag{H.96}$$

   and then use the identities $\langle 2^- | \not\epsilon_3(2) = 0$ and $\epsilon^+_3(2)\epsilon^-_4(3) = 0$. We see that only one of the non-Abelian pieces survives and we have

$$
\begin{aligned}
iA_4(1^+_{\bar{q}},2^-_q,3^+_g,4^-_g) &= \left(\frac{i}{\sqrt{2}}\right)^2 \frac{-i}{s_{12}} 2 \left\langle 2^- \middle| \not{\epsilon}^-_4(3)(p_4 \cdot \epsilon^+_3(2)) \middle| 1^- \right\rangle \\
&= \frac{i}{s_{12}} \left\langle 2^- \right| \frac{\langle 24\rangle[43]}{\sqrt{2}\langle 23\rangle} \frac{-\sqrt{2}}{[34]} \left|4^+\right\rangle [31] \\
&= i\frac{\langle 24\rangle^2[31]}{\langle 12\rangle[21]\langle 23\rangle} \,.
\end{aligned} \tag{H.97}
$$

To rewrite everything in terms of right-handed spinor product we multiply and divide by $\langle 24\rangle$ and use momentum conservation $-[12]\langle 24\rangle = [13]\langle 34\rangle$ to obtain

$$
\begin{aligned}
iA_4(1^+_{\bar{q}},2^-_q,3^+_g,4^-_g) &= i\frac{\langle 24\rangle^3[31]}{\langle 12\rangle\langle 23\rangle[13]\langle 34\rangle} \\
&= i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \,.
\end{aligned} \tag{H.98}
$$

We now repeat the computation with

$$
\epsilon^+_{3,\mu}(4) = \frac{\langle 4^-|\,\gamma_\mu\,|3^-\rangle}{\sqrt{2}\langle 43\rangle} \,, \tag{H.99}
$$

and use eq.(H.53), $p_4\cdot\epsilon^+_3(4) = p_3\cdot\epsilon^-_4(3) = 0$, and eq.(H.57), $\epsilon^+_3(4)\cdot\epsilon^-_4(3) = 0$, so the non-Abelian piece vanishes completely. We are left with

$$
\begin{aligned}
iA_4(1^+_{\bar{q}},2^-_q,3^+_g,4^-_g) &= \left(\frac{i}{\sqrt{2}}\right)^2 \left\langle 2^- \middle| \not{\epsilon}^+_3(4) \frac{i(\not{p}_2+\not{p}_3)}{s_{23}} \not{\epsilon}^-_4(3) \middle| 1^- \right\rangle \\
&= \left(\frac{i}{\sqrt{2}}\right)^2 \frac{i}{s_{23}} \left\langle 2^- \right| \frac{\sqrt{2}}{\langle 43\rangle} \left|4^+\right\rangle \left\langle 3^+ \middle| 2^- \right\rangle \left\langle 2^- \right| \frac{-\sqrt{2}}{[34]} \left|4^+\right\rangle \left\langle 3^+ \middle| 1^- \right\rangle \\
&= i\frac{\langle 24\rangle^2[32][31]}{\langle 23\rangle[32]\langle 34\rangle[43]} \\
&= i\frac{\langle 24\rangle^3[31]}{\langle 23\rangle\langle 34\rangle\langle 21\rangle[31]} \\
&= i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \,,
\end{aligned} \tag{H.100}
$$

where we used momentum conservation $\langle 24\rangle[43] = -\langle 21\rangle[13]$.

3. For 3^-4^+, choose $\epsilon^-_\mu(3,4)$ and $\epsilon^+_\mu(4,3)$.

Solution. We have the polarization vectors

$$
\epsilon^-_{3,\mu}(4) = -\frac{\langle 4^+|\,\gamma_\mu\,|3^+\rangle}{\sqrt{2}[43]} \,, \tag{H.101}
$$

$$
\epsilon^+_{4,\mu}(3) = \frac{\langle 3^-|\,\gamma_\mu\,|4^-\rangle}{\sqrt{2}\langle 34\rangle} \,. \tag{H.102}
$$

and use $p_4 \cdot \epsilon_3^-(4) = p_3 \cdot \epsilon_4^+(3) = 0$ and $\epsilon_3^-(4) \cdot \epsilon_4^+(3) = 0$, so the non-Abelian piece vanishes completely. We get

$$\begin{aligned}
iA_4(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-) &= \left(\frac{i}{\sqrt{2}}\right)^2 \langle 2^- | \not{\epsilon}_3^-(4) \frac{i(\not{p}_2 + \not{p}_3)}{s_{23}} \not{\epsilon}_4^+(3) | 1^- \rangle \\
&= \left(\frac{i}{\sqrt{2}}\right)^2 \frac{i}{s_{23}} \langle 2^- | \frac{-\sqrt{2}}{[43]} | 3^+ \rangle [42] \langle 2^- | \frac{\sqrt{2}}{\langle 34 \rangle} | 3^+ \rangle [41] \\
&= i \frac{\langle 23 \rangle^2 [42][41]}{\langle 23 \rangle [32] \langle 34 \rangle [43]} \\
&= i \frac{\langle 23 \rangle^3 [42][41]}{\langle 23 \rangle [32] \langle 34 \rangle [41] \langle 12 \rangle} \\
&= i \frac{\langle 23 \rangle^3 \langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} ,
\end{aligned} \tag{H.103}$$

where we used $-[43]\langle 32 \rangle = [41]\langle 12 \rangle$ and then $\langle 13 \rangle [32] = -\langle 14 \rangle [42]$.

H.13 Squared amplitude for $q\bar{q} \to gg$

The non-vanishing helicity configurations for $q\bar{q} \to gg$ are

$$M\left(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-\right) = g^2 \left[\frac{\langle 24 \rangle^3 \langle 14 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} T^{a_3} T^{a_4} + \frac{\langle 24 \rangle^3 \langle 14 \rangle}{\langle 12 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} T^{a_4} T^{a_3} \right] , \tag{H.104}$$

$$M\left(1_{\bar{q}}^+, 2_q^-, 3_g^-, 4_g^+\right) = g^2 \left[\frac{\langle 23 \rangle^3 \langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} T^{a_3} T^{a_4} + \frac{\langle 23 \rangle^3 \langle 13 \rangle}{\langle 12 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} T^{a_4} T^{a_3} \right] , \tag{H.105}$$

with the other amplitudes given by parity flip. Square the amplitudes and compute the sum over helicities $\sum_{hel} |M\left(1_{\bar{q}}, 2_q, 3_g, 4_g\right)|^2$.

Solution. We fix

$$a_1 = \frac{\langle 24 \rangle^3 \langle 14 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} , \tag{H.106}$$

$$a_2 = \frac{\langle 24 \rangle^3 \langle 14 \rangle}{\langle 12 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} , \tag{H.107}$$

such that

$$M\left(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-\right) = g^2 \left(a_1 T^{a_3} T^{a_4} + a_2 T^{a_4} T^{a_3}\right) . \tag{H.108}$$

Then

$$\sum_{\text{colour}} |M\left(1_{\bar{q}}^+, 2_q^-, 3_g^+, 4_g^-\right)|^2 = g^4 \left[\left(|a_1|^2 + |a_2|^2\right) \mathrm{tr}(T^{a_3} T^{a_4} T^{a_4} T^{a_3}) + (a_1^* a_2 + a_1 a_2^*) \mathrm{tr}(T^{a_3} T^{a_4} T^{a_3} T^{a_4}) \right] , \tag{H.109}$$

and the colour traces are

$$\begin{aligned}\mathrm{tr}(T^{a_3}T^{a_4}T^{a_4}T^{a_3}) &= \frac{N_c^2-1}{N_c}\,\mathrm{tr}(T^{a_3}T^{a_3}) \\ &= \frac{(N_c^2-1)^2}{N_c}\,. \end{aligned} \tag{H.110}$$

Using the Fierz identity for $SU(N_c)$ matrices,

$$(T^c)_{i_1}^{j_1}(T^c)_{i_2}^{j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N_c}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\,, \tag{H.111}$$

we can write

$$\begin{aligned}\mathrm{tr}(T^{a_3}T^{a_4}T^{a_3}T^{a_4}) &= (\mathrm{tr}(T^a))^2 - \frac{1}{N_c}\,\mathrm{tr}(T^{a_4}T^{a_4}) \\ &= -\frac{N_c^2-1}{N_c}\,. \end{aligned} \tag{H.112}$$

We have

$$|a_1|^2 = \frac{s_{24}^3}{s_{12}s_{23}s_{34}} = \frac{s_{24}^3}{s_{12}^2 s_{23}}\,, \tag{H.113}$$

$$|a_2|^2 = \frac{s_{24}^2 s_{14}}{s_{12}s_{34}s_{13}} = \frac{s_{24}s_{14}}{s_{12}^2}\,, \tag{H.114}$$

$$\begin{aligned} a_1 a_2^* &= \frac{\langle 24\rangle^3\langle 14\rangle[41][42]^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle[21][42][34][13]} \\ &= -\frac{s_{24}^3 s_{14}}{s_{12}s_{34}\langle 23\rangle\langle 41\rangle[42][13]}\,, \end{aligned} \tag{H.115}$$

and

$$\begin{aligned} a_1 a_2^* + a_1^* a_2 &= \frac{s_{24}^3 s_{14}}{s_{12}^2}\left(\frac{1}{\langle 23\rangle\langle 14\rangle[42][13]} + \frac{1}{[32][41]\langle 24\rangle\langle 31\rangle}\right) \\ &= \frac{s_{24}^3 s_{14}}{s_{12}^2}\frac{[32][41]\langle 24\rangle\langle 31\rangle + \langle 23\rangle\langle 14\rangle[42][13]}{s_{23}s_{14}s_{24}s_{13}} \\ &= -\frac{s_{24}^3 s_{14}}{s_{12}^2}\frac{\langle 23\rangle[31][32]\langle 31\rangle + \langle 13\rangle[32]\langle 23\rangle[13]}{s_{23}s_{14}s_{24}s_{13}} \\ &= \frac{s_{24}^3 s_{14}}{s_{12}^2}\frac{2s_{13}s_{23}}{s_{23}s_{14}s_{24}s_{13}} \\ &= 2\frac{s_{24}^2}{s_{12}^2}\,, \end{aligned} \tag{H.116}$$

where we used momentum conservation $\langle 24\rangle[41] = -\langle 23\rangle[31]$. As before, we can use the shorthand

$$s = s_{12}\,, \qquad t = s_{13} = s_{24}\,, \qquad u = s_{14} = s_{23}\,, \tag{H.117}$$

to obtain the squared amplitude

$$\begin{aligned}\sum_{\text{col}}|M\left(1^+_{\bar{q}},2^-_q,3^+_g,4^-_g\right)|^2 &= g^4\left[\left(\frac{t^3}{s^2u}+\frac{tu}{s^2}\right)\frac{\left(N_c^2-1\right)^2}{N_c}-2\frac{t^2}{s^2}\frac{N_c^2-1}{N_c}\right] \\ &= g^4\left[\left(t^2+u^2\right)\frac{t}{s^2u}\frac{\left(N_c^2-1\right)^2}{N_c}-2\frac{t^2}{s^2}\frac{N_c^2-1}{N_c}\right] \\ &= g^4\left[\left(\frac{t}{u}-2\frac{t^2}{s^2}\right)\frac{\left(N_c^2-1\right)^2}{N_c}-2\frac{t^2}{s^2}\frac{N_c^2-1}{N_c}\right] \\ &= g^4\left[\frac{t}{u}\frac{\left(N_c^2-1\right)^2}{N_c}-2\frac{t^2}{s^2}N_c\left(N_c^2-1\right)\right], \end{aligned} \tag{H.118}$$

where we used $s^2=(t^2+u^2)=t^2+u^2+2ut$ on the third line.

$M\left(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g\right)$ is obtained from $M\left(1^+_{\bar{q}},2^-_q,3^+_g,4^-_g\right)$ by swapping labels 3 and 4, thus the squared matrix elements are related by swapping the u- and t-channel,

$$\sum_{\text{col}}|M\left(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g\right)|^2 = g^4\left[\frac{u}{t}\frac{\left(N_c^2-1\right)^2}{N_c}-2\frac{u^2}{s^2}N_c\left(N_c^2-1\right)\right]. \tag{H.119}$$

The other amplitudes are given by parity flip,

$$\sum_{\text{col}}|M\left(1^-_{\bar{q}},2^+_q,3^-_g,4^+_g\right)|^2 = \sum_{\text{col}}|M\left(1^+_{\bar{q}},2^-_q,3^+_g,4^-_g\right)|^2, \tag{H.120}$$

$$\sum_{\text{col}}|M\left(1^-_{\bar{q}},2^+_q,3^+_g,4^-_g\right)|^2 = \sum_{\text{col}}|M\left(1^+_{\bar{q}},2^-_q,3^-_g,4^+_g\right)|^2, \tag{H.121}$$

so

$$\sum_{\text{col,hel}}|M\left(1_{\bar{q}},2_q,3_g,4_g\right)|^2 = g^4\left[2\frac{(N_c^2-1)^2}{N_c}\frac{t^2+u^2}{tu}-4N_c(N_c^2-1)\frac{t^2+u^2}{s^2}\right]. \tag{H.122}$$

H.14 Four-Gluon Amplitude

Compute the four-gluon sub-amplitudes,

1. $A_4^{tree}(1^-,2^-,3^+,4^+)$

   Hint. Use the polarizations $\epsilon^-(1,3)$, $\epsilon^-(2,3)$, $\epsilon^+(3,2)$ and $\epsilon^-(4,2)$.

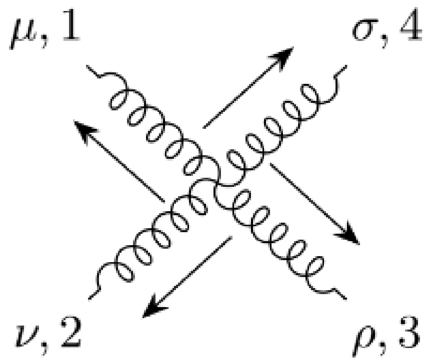
   Solution. We use the identities between polarization vectors,

$$\begin{aligned} &\epsilon^-(1,3)\epsilon^-(2,3)=0, \qquad \epsilon^+(3,2)\epsilon^+(4,2)=0, \qquad \epsilon^+(3,2)\epsilon^-(2,3)=0, \\ &\epsilon^+(3,2)\epsilon^-(1,3)=0, \qquad \epsilon^+(4,2)\epsilon^-(2,3)=0, \end{aligned} \tag{H.123}$$

   so the only non-vanishing scalar product between polarization is $\epsilon^+(4,2)\epsilon^-(1,3)$ and we have further

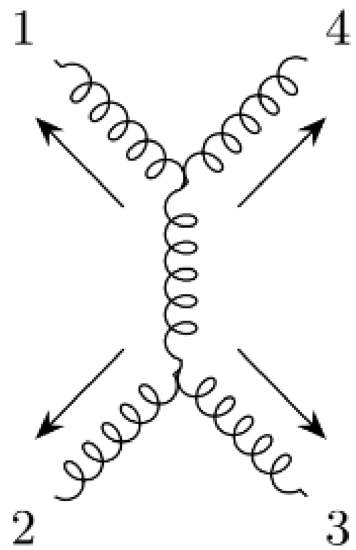
$$\epsilon^{\pm}(p_i, k) \cdot p_i = \epsilon^{\pm}(p_i, k) \cdot k = 0, \quad \forall i\,. \tag{H.124}$$

Let us compute the Feynman diagrams using the colour-ordered Feynman rules,



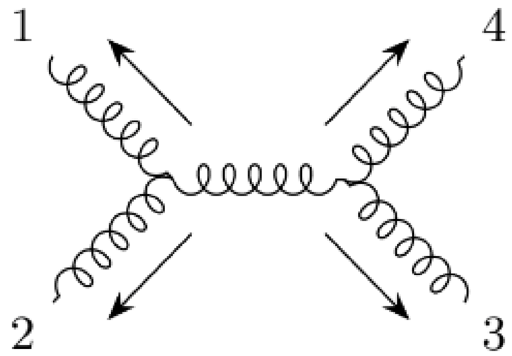
$$\begin{aligned} iA_a &= \frac{i}{2}\left(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}\right)\epsilon^-_\mu(1,3)\epsilon^-_\nu(2,3)\epsilon^+_\rho(3,2)\epsilon^+_\sigma(4,2) \\ &= 0\,. \end{aligned} \tag{H.125}$$

We have



$$
\begin{aligned}
iA_b &= \left(\frac{i}{\sqrt{2}}\right)^2 \frac{-i}{s_{14}} \epsilon^{-\mu}(1,3)\epsilon^{-\nu}(2,3)\epsilon^{+\rho}(3,2)\epsilon^{+\sigma}(4,2) \\
&\cdot \left(\cancelto{0}{\left(2p_1 + p_4\right)_\sigma} g_\mu{}^\alpha - \cancelto{0}{\left(p_1 + 2p_4\right)_\mu} g_\sigma{}^\alpha + \left(p_4 - p_1\right)^\alpha g_{\mu\sigma} \right) \\
&\cdot \left(\cancelto{0}{-(2p_2 + p_3)_\rho g_{\nu\alpha}} - (p_2 - p_3)_\alpha \cancelto{0}{g_{\nu\rho}} + \cancelto{0}{(2p_3 - p_2)_\nu g_{\rho\alpha}} \right) \\
&= 0\,.
\end{aligned}
\tag{H.126}
$$

The non-vanishing contribution is



$$
\begin{aligned}
iA_c =& \left(\frac{i}{\sqrt{2}}\right)^2 \frac{-i}{s_{12}} \epsilon^{-\mu}(1,3)\epsilon^{-\nu}(2,3)\epsilon^{+\rho}(3,2)\epsilon^{+\sigma}(4,2) \\
&\cdot \left((p_1-p_4)^\alpha \cancelto{0}{g_{\mu\nu}} + \cancelto{0}{\left(2p_2+p_1\right)_\mu} \cancelto{0}{g_\nu{}^\alpha} - \cancelto{0}{\left(2p_1+p_2\right)_\nu} g_\mu{}^\alpha \right) \\
&\cdot \left((p_3-p_4)_\alpha \cancelto{0}{g_{\rho\sigma}} + \cancelto{0}{\left(2p_4+p_3\right)_\rho} g_{\sigma\alpha} - \cancelto{0}{\left(2p_3+p_4\right)_\sigma} \cancelto{0}{g_{\rho\alpha}} \right) \\
=& -4\left(\frac{i}{\sqrt{2}}\right)^2 \frac{-i}{s_{12}} (\epsilon^-(2,3)\cdot p_1)(\epsilon^+(3,2)\cdot p_4)(\epsilon^-(1,3)\cdot\epsilon^+(4,2)) \\
=& -\frac{2i}{s_{12}} \left(-\frac{\langle 3^+|\not{p}_1|2^+\rangle}{\sqrt{2}[32]}\right) \frac{\langle 2^-|\not{p}_4|3^-\rangle}{\sqrt{2}\langle 23\rangle} \left(-\frac{\langle 3^+|\gamma_\mu|1^+\rangle}{\sqrt{2}[31]} \frac{\langle 2^-|\gamma^\mu|4^-\rangle}{\sqrt{2}\langle 24\rangle}\right) \\
=& -\frac{2i}{s_{12}} \frac{\cancel{[31]}\langle 12\rangle\cancel{\langle 24\rangle}[43]2\langle 21\rangle[34]}{4[32]\langle 23\rangle\cancel{[31]}\cancel{\langle 24\rangle}} \\
=& -\frac{i}{s_{34}} \frac{\langle 12\rangle^2[34]^2}{\langle 23\rangle[32]} \\
=& \, i\frac{\langle 12\rangle^2[34]}{\langle 23\rangle[32]\langle 34\rangle} \qquad (\star) \\
=& \, i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\
=& \, i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \,.
\end{aligned}
\tag{H.127}
$$

where in $(\star)$ of eq. (H.127) we multiplied and divided by $\langle 21\rangle$ and use momentum conservation $[32]\langle 21\rangle = -[34]\langle 41\rangle$. Note that in $(\star)$ (H.127) we could also multiply and divide by $[41]^2$ and use momentum conservation $[41]\langle 12\rangle = -[43]\langle 32\rangle$ and $\langle 34\rangle[41] = -\langle 32\rangle[21]$ to obtain

$$
\begin{aligned}
iA_c &= -i\frac{[34]^3\langle 32\rangle^2}{\langle 23\rangle[32]\langle 32\rangle[21][41]} \\
&= i\frac{[34]^4}{[12][23][34][41]} \,.
\end{aligned}
\tag{H.128}
$$

Since the diagram with the crossed gluon legs does not contribute to the color ordered amplitude, we have

$$
iA_4^{\text{tree}}(1^-,2^-,3^+,4^+) = iA_c \,. \tag{H.129}
$$

2. $A_4^{tree}(1^-,2^+,3^-,4^+)$

   Hint Use the photon decoupling identity for gluon 1.

   Solution. The photon decoupling identity for gluon 1 is

$$
A_4^{\text{tree}}(1,2,3,4) + A_4^{\text{tree}}(1,3,4,2) + A_4^{\text{tree}}(1,4,2,3) = 0 \,. \tag{H.130}
$$

thus we can write (using cyclicity)

$$
\begin{aligned}
A_4^{\text{tree}}(1^-,2^+,3^-,4^+) &= -A_4^{\text{tree}}(1^-,3^-,4^+,2^+) - A_4^{\text{tree}}(3^-,1^-,4^+,2^+) \\
&= -\left(\frac{\langle 13\rangle^3}{\langle 34\rangle\langle 42\rangle\langle 21\rangle} + \frac{\langle 31\rangle^3}{\langle 14\rangle\langle 42\rangle\langle 23\rangle}\right) \\
&= -\frac{\langle 13\rangle^3}{\langle 42\rangle}\left(\frac{1}{\langle 34\rangle\langle 21\rangle} - \frac{1}{\langle 14\rangle\langle 23\rangle}\right) \\
&= -\frac{\langle 13\rangle^3}{\langle 42\rangle}\left(\frac{\langle 14\rangle\langle 23\rangle + \langle 34\rangle\langle 12\rangle}{\langle 34\rangle\langle 21\rangle\langle 14\rangle\langle 23\rangle}\right) \\
&= \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,,
\end{aligned}
\tag{H.131}
$$

where we used the Schouten identity $\langle 12\rangle\langle 34\rangle + \langle 14\rangle\langle 23\rangle + \langle 13\rangle\langle 42\rangle = 0$. So we get for the MHV amplitude,

$$
A_4^{\text{tree}}(1,2,3,4) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{H.132}
$$

where i and j are the two negative helicity gluons.

H.15 Kleiss-Kuijf Relation

Write the Kleiss-Kuiff relation

$$
A_n^{tree}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in\{\alpha\}\sqcup\{\beta^T\}} A_n^{tree}\left(1,\sigma(\{\alpha\}\{\beta^T\}),n\right)\,, \tag{H.133}
$$

for

1. $A(12534)$ with $\{\alpha\}=\{2\}$, $\{\beta\}=\{3,4\}$,

   Solution. $A(12534)$ with $\{\alpha\}=\{2\}$, $\{\beta\}=\{3,4\}$ has $\{\alpha\}\cup\{\beta^T\}=\{2,4,3\}$. So

   $$
   \{\alpha\}\sqcup\{\beta^T\} = \{(243),(423),(432)\}\,, \tag{H.134}
   $$

   and

   $$
   A(12534) = A(12435) + A(14235) + A(14325)\,. \tag{H.135}
   $$

2. $A(12354)$ with $\{\alpha\}=\{2,3\}$, $\{\beta\}=\{4\}$,

   Solution. $A(12354)$ with $\{\alpha\}=\{2,3\}$, $\{\beta\}=\{4\}$ yields

   $$
   A(12354) = -A(12345) - A(12435) - A(14235)\,. \tag{H.136}
   $$

3. $A(123654)$ with $\{\alpha\} = \{2,3\}$, $\{\beta\} = \{5,4\}$.

   Solution. $A(123654)$ with $\{\alpha\} = \{2,3\}$, $\{\beta\} = \{5,4\}$ has $\{\alpha\} \cup \{\beta^T\} = \{2,3,4,5\}$ and yields

$$A(123654) = A(123456) + A(124356) + A(124536) + A(142356) + A(142536) + A(145236)\,. \tag{H.137}$$

H.16 Four-gluon scattering: multiperipheral decomposition

Consider the scattering process $gg \to gg$ and

1. Compute $M_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$ using the multiperipheral-based colour decomposition.

   Hint Use $f^{abc}f^{abd} = N_c\delta^{cd}$ and $f^{a_1a_2c}f^{ca_3a_4}f^{da_3a_1}f^{da_4a_2} = \dfrac{N_c^2(N_c^2-1)}{2}$.

   Solution. We have

$$\begin{aligned} M_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) &= g^2[F^{a_1a_2c}F^{ca_3a_4}A(1^-, 2^-, 3^+, 4^+) + F^{a_1a_3c}F^{ca_2a_4}A(1^-, 3^+, 2^-, 4^+)] \\ &= -2g^2[\underbrace{f^{a_1a_2c}f^{ca_3a_4}}_{=:c_1}\underbrace{A(1^-, 2^-, 3^+, 4^+)}_{=:a_1^{--++}} + \underbrace{f^{a_1a_3c}f^{ca_2a_4}}_{=:c_2}\underbrace{A(1^-, 3^+, 2^-, 4^+)}_{=:a_2^{--++}}]\,, \end{aligned} \tag{H.138}$$

   where we used $F^{abc} = i\sqrt{2}f^{abc}$.

2. Square the amplitude.

   Hint Use $f^{abc}f^{abd} = N_c\delta^{cd}$ and $f^{a_1a_2c}f^{ca_3a_4}f^{da_3a_1}f^{da_4a_2} = \frac{N_c^2(N_c^2-1)}{2}$.

   Solution. To square the amplitude we need to compute the colour-factors, which come from the sum over colour. We have $c_i^* = c_i$ since $f^{abc*} = f^{abc}$, and furthermore $c_1c_1^* = c_2c_2^* = c_1c_1$, since it is simply a relabelling of the summation indices. We compute

$$\begin{aligned} c_1c_1 &= (f^{a_1a_2c}f^{a_1a_2d})(f^{a_3a_4c}f^{a_3a_4d}) \\ &= N_c\delta_c^c \\ &= N_c^2\left(N_c^2-1\right)\,, \end{aligned} \tag{H.139}$$

$$\begin{aligned} c_1c_2 &= f^{a_1a_2c}f^{ca_3a_4}f^{a_1a_3d}f^{da_2a_4} \\ &= \frac{N_c^2\,(N_c^2-1)}{2}\,. \end{aligned} \tag{H.140}$$

   So the squared amplitude reads

$$\begin{aligned} |M_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)|^2 = 4g^4\Big[&N_c^2\left(N_c^2-1\right)\left(|a_1^{--++}|^2 + |a_2^{--++}|^2\right) \\ &+ \frac{N_c^2\,(N_c^2-1)}{2}\left(a_1^{--++}(a_2^{--++})^* + a_2^{--++}(a_1^{--++})^*\right)\Big]\,, \end{aligned} \tag{H.141}$$

with

$$a_1^{--++} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{H.142}$$

$$a_2^{--++} = \frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}\,, \tag{H.143}$$

we have

$$\begin{aligned} a_1^{--++}(a_2^{--++})^* &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[21]^4}{[31][23][43][14]} \\ &= -\frac{s_{12}^4}{s_{14}s_{23}\langle 12\rangle\langle 34\rangle[31][42]} \\ &= \frac{s_{12}^4}{s_{13}s_{34}} \\ &= \frac{s_{12}^3}{s_{14}^2 s_{13}}\,, \end{aligned} \tag{H.144}$$

where we used momentum conservation $[42]\langle 21\rangle = -[43]\langle 31\rangle$ and all the other contributions can be computed in complete analogy,

$$|a_1^{--++}|^2 = \frac{s_{12}^2}{s_{14}^2} \tag{H.145}$$

$$|a_2^{--++}|^2 = \frac{s_{12}^4}{s_{14}^2 s_{24}^2}\,. \tag{H.146}$$

3. Compute the helicity configurations and sum over helicities.

Solution. To compute the sum over helicities we use that only amplitudes with two negative helicities are non-vanishing. Furthermore, due to parity we have

$$|M_4^{\rm tree}(1^-,2^-,3^+,4^+)|^2 = |M_4^{\rm tree}(1^+,2^+,3^-,4^-)|^2\,, \tag{H.147}$$

$$|M_4^{\rm tree}(1^-,2^+,3^+,4^-)|^2 = |M_4^{\rm tree}(1^+,2^-,3^-,4^+)|^2\,, \tag{H.148}$$

$$|M_4^{\rm tree}(1^+,2^-,3^+,4^-)|^2 = |M_4^{\rm tree}(1^-,2^+,3^-,4^+)|^2\,. \tag{H.149}$$

So the squared matrix element, summed over colour and helicities reads

$$|\mathcal{M}|^2 = 2\Big(|M_4^{\rm tree}(1^-,2^-,3^+,4^+)|^2 + |M_4^{\rm tree}(1^+,2^-,3^-,4^+)|^2 + |M_4^{\rm tree}(1^-,2^+,3^-,4^+)|^2\Big). \tag{H.150}$$

The first two summands only need one computation due to the cyclic invariance of the matrix element,

$$|M_4^{\rm tree}(1^+,2^-,3^-,4^+)|^2 = |M_4^{\rm tree}(1^-,2^-,3^+,4^+)|^2\Big|_{\substack{1\to 2\\ 4\to 1\\ 2\to 3\\ 3\to 4}}\,. \tag{H.151}$$

To express everything in terms of scalar products we use the partial amplitudes (H.138),

$$a_1^{-+-+} = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{H.152}$$

$$a_2^{-+-+} = \frac{\langle 13\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}\,, \tag{H.153}$$

with

$$a_1^{-+-+}(a_2^{-+-+})^* = \frac{s_{13}^3}{s_{14}^2 s_{12}}\,, \tag{H.154}$$

$$|a_1^{-+-+}|^2 = \frac{s_{24}^4}{s_{12}^2 s_{14}^2}\,, \tag{H.155}$$

$$|a_2^{-+-+}|^2 = \frac{s_{24}^2}{s_{14}^2}\,. \tag{H.156}$$

So our squared matrix element reads

$$\begin{aligned}
|\mathcal{M}|^2 &= 2\Big(|M_4^{\text{tree}}(1^-,2^-,3^+,4^+)|^2+|M_4^{\text{tree}}(1^+,2^-,3^-,4^+)|^2+|M_4^{\text{tree}}(1^-,2^+,3^-,4^+)|^2\Big)\\
&= 8g^4N_c^2\left(N_c^2-1\right)\Big[\big(\frac{s_{12}^2}{s_{14}^2}+\frac{s_{12}^4}{s_{14}^2 s_{24}^2}+\frac{s_{12}^3}{s_{14}^2 s_{13}}\big)+\big(\frac{s_{23}^2}{s_{21}^2}+\frac{s_{23}^4}{s_{21}^2 s_{31}^2}+\frac{s_{23}^3}{s_{21}^2 s_{24}}\big)\\
&\quad+\big(\frac{s_{24}^4}{s_{12}^2 s_{14}^2}+\frac{s_{24}^2}{s_{14}^2}+\frac{s_{13}^3}{s_{14}^2 s_{12}}\big)\Big]\\
&= 8g^4N_c^2\left(N_c^2-1\right)\left[\frac{2\left(s_{12}^2+s_{13}s_{12}+s_{13}^2\right)^3}{s_{12}^2 s_{13}^2\left(s_{12}+s_{13}\right)^2}\right]\\
&= 8g^4N_c^2\left(N_c^2-1\right)\frac{\left(s_{12}^2+s_{13}^2+s_{14}^2\right)^3}{4s_{12}^2 s_{13}^2 s_{14}^2}\,,
\end{aligned} \tag{H.157}$$

where we first wrote everything in terms of the independent kinematic invariants s_{12} and s_{13} and then used $s_{12}+s_{13} = -s_{14}$ to highlight the singularity structure. To obtain the fully differential cross section we have to average over the colour and helicities of the initial states by multiplying $(1/2)^2(1/(N_c^2-1))^2$. One can easily verify that our result is the same as

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4(N_c^2-1)^2}|\mathcal{M}|^2 = \frac{4N_c^2}{N_c^2-1}g^4\left[3-\frac{s_{12}s_{13}}{s_{14}^2}-\frac{s_{12}s_{14}}{s_{13}^2}-\frac{s_{13}s_{14}}{s_{12}^2}\right]\,, \tag{H.158}$$

which can be found in reviews of two-jet production at hadron colliders.

H.17 Four-gluon scattering: trace-based decomposition

Consider the scattering process $gg \to gg$ and

1. Compute $M_4^{\text{tree}}(1,2,3,4)$ using the trace-based colour decomposition.

2. Square the amplitude and compute $|M_4^{\text{tree}}(1^-,2^-,3^+,4^+)|^2$

Hint Remember that you can use the $U(N_c)$ completeness relation for gluon amplitudes.

Solution. In the following we use the shorthand notation $T^{a_1}\dots T^{a_n} = T^{1\dots n}$. The colour ordered amplitude may be written, after applying reflection symmetry, as

$$\begin{aligned} M_4^{\text{tree}}(1,2,3,4) &= g^2 A(1234)[\,\mathrm{tr}\big(T^{1234}\big) + \mathrm{tr}\big(T^{1432}\big)] + g^2 A(1324)[\,\mathrm{tr}\big(T^{1324}\big) + \mathrm{tr}\big(T^{1423}\big)] \\ &+ g^2 A(1342)[\,\mathrm{tr}\big(T^{1342}\big) + \mathrm{tr}\big(T^{1243}\big)]\,. \end{aligned} \tag{H.159}$$

This can be further simplified by using the photon decoupling identity (replace e.g. T^{a_4} by the $U(1)$ generator, $\mathbb{1}$)

$$A(1342) = -A(1234) - A(1324)\,. \tag{H.160}$$

which yields

$$\begin{aligned} M_4^{\text{tree}}(1,2,3,4) &= g^2 A(1234)\underbrace{[\,\mathrm{tr}\big(T^{1234}\big) + \mathrm{tr}\big(T^{1432}\big) - \mathrm{tr}\big(T^{1342}\big) - \mathrm{tr}\big(T^{1243}\big)]}_{c_1} \\ &+ g^2 A(1324)\underbrace{[\,\mathrm{tr}\big(T^{1324}\big) + \mathrm{tr}\big(T^{1423}\big) - \mathrm{tr}\big(T^{1342}\big) - \mathrm{tr}\big(T^{1243}\big)]}_{c_2}\,. \end{aligned} \tag{H.161}$$

Instead of rewriting everything in terms of the structure constants, we want to compute the color factors directly from the trace decomposition. We use $\mathrm{tr}(T^{1\dots n})^* = \mathrm{tr}(T^{n\dots 1})$. We saw, that the color algebra can be performed for $U(N_c)$, that means we use the completeness relation,

$$(T^a)^i{}_j (T^a)^k{}_l = \delta^i_l \delta^k_j\,. \tag{H.162}$$

Since the generators of $U(N_c)$ are hermitian but not traceless we have

$$\mathrm{tr}(T^a) \neq 0 \qquad\qquad \delta^{aa} = N_c^2\,. \tag{H.163}$$

To perform the colour algebra, we use

$$\mathrm{tr}(AT^aB)\,\mathrm{tr}(CT^aD) = \mathrm{tr}(ADCB)\,, \tag{H.164}$$

and

$$\mathrm{tr}(AT^aBT^aC)|_{B\neq\mathbb{1}} = \mathrm{tr}(AC)\,\mathrm{tr}(B)\,, \tag{H.165}$$

for A, B, C, D being strings of generators of $U(N_c)$. Let us e.g. look at

$$\begin{aligned}\mathrm{tr}\big(T^{1234}\big)(\mathrm{tr}\big(T^{1432}\big))^* &= \mathrm{tr}\big(T^{1234}\big)\,\mathrm{tr}\big(T^{2341}\big)\\ &= \mathrm{tr}\big(T^{234}T^{234}\big)\\ &= \mathrm{tr}\big(T^{34}\big)\,\mathrm{tr}\big(T^{34}\big)\\ &= \mathrm{tr}\big(T^{4}T^{4}\big)\\ &= \delta^{a_4 a_4}\\ &= N_c^2\,. \end{aligned} \tag{H.166}$$

So, computing the colour factors boils down to applying the two identities (which is completely algorithmic) and gives

$$\begin{aligned}
&\mathrm{tr}\left(T^{1,2,3,4}\right)^2 = N_c^2, && \mathrm{tr}\left(T^{1,2,3,4}\right)\mathrm{tr}\left(T^{1,2,4,3}\right) = N_c^2, && \mathrm{tr}\left(T^{1,2,4,3}\right)^2 = N_c^2,\\
&\mathrm{tr}\left(T^{1,2,3,4}\right)\mathrm{tr}\left(T^{1,3,2,4}\right) = N_c^2, && \mathrm{tr}\left(T^{1,2,4,3}\right)\mathrm{tr}\left(T^{1,3,2,4}\right) = N_c^2, && \mathrm{tr}\left(T^{1,2,3,4}\right)\mathrm{tr}\left(T^{1,3,4,2}\right) = N_c^2,\\
&\mathrm{tr}\left(T^{1,2,4,3}\right)\mathrm{tr}\left(T^{1,3,4,2}\right) = N_c^4, && \mathrm{tr}\left(T^{1,3,2,4}\right)\mathrm{tr}\left(T^{1,3,4,2}\right) = N_c^2, && \mathrm{tr}\left(T^{1,3,4,2}\right)^2 = N_c^2,\\
&\mathrm{tr}\left(T^{1,2,3,4}\right)\mathrm{tr}\left(T^{1,4,2,3}\right) = N_c^2, && \mathrm{tr}\left(T^{1,2,4,3}\right)\mathrm{tr}\left(T^{1,4,2,3}\right) = N_c^2, && \mathrm{tr}\left(T^{1,3,4,2}\right)\mathrm{tr}\left(T^{1,4,2,3}\right) = N_c^2,\\
&\mathrm{tr}\left(T^{1,2,3,4}\right)\mathrm{tr}\left(T^{1,4,3,2}\right) = N_c^4, && \mathrm{tr}\left(T^{1,2,4,3}\right)\mathrm{tr}\left(T^{1,4,3,2}\right) = N_c^2, && \mathrm{tr}\left(T^{1,3,2,4}\right)\mathrm{tr}\left(T^{1,4,3,2}\right) = N_c^2,\\
&\mathrm{tr}\left(T^{1,3,4,2}\right)\mathrm{tr}\left(T^{1,4,3,2}\right) = N_c^2, && \mathrm{tr}\left(T^{1,4,2,3}\right)\mathrm{tr}\left(T^{1,4,3,2}\right) = N_c^2, && \mathrm{tr}\left(T^{1,4,3,2}\right)^2 = N_c^2\,.
\end{aligned} \tag{H.167}$$

We get, as before,

$$c_1 c_1^* = 4N_c^2(-1+N_c^2), \qquad c_1 c_2^* = 2N_c^2(-1+N_c^2)\,. \tag{H.168}$$

The rest is completely analogous to the multiperipheral decomposition.

H.18 Multi-Regge Kinematics (MRK)

Consider the scattering of $2 \to (n-2)$ gluons $g(-p_1)g(-p_n) \to g(p_2)\dots g(p_{n-1})$.

In MRK: $p^+ \gg \dots \gg p_{n-1}^+$, $p_2^- \ll \dots \ll p_{n-1}^-$.

1. For the helicity configuration $(1^-, 2^+, 3^+, \dots, (n-1)^+, n^-)$, compute the sub-amplitude $A\,(1^-, 2^+, 3^+, \dots, (n-1)^+, n^-)$.

Solution.

$$
\begin{aligned}
A\left(1^-,2^+,3^+,\ldots,(n-1)^+,n^-\right) &= \frac{\langle 1n\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle (n-1)n\rangle\langle n1\rangle} \\
&= \frac{s^2}{ip_{2\perp}\sqrt{\frac{-p_1^+}{\cancel{p_2^+}}}\,p_{3\perp}\sqrt{\frac{\cancel{p_2^+}}{\cancel{p_3^+}}}\ldots p_{n-1\perp}\sqrt{\frac{\cancel{p_{n-2}^+}}{\cancel{p_{n-1}^+}}}\,i\sqrt{-p_n^-\cancel{p_{n-1}^+}}\sqrt{s}} \\
&= -\frac{s}{p_{2\perp}p_{3\perp}\cdots p_{n-1\perp}}\,. \qquad \text{(H.169)}
\end{aligned}
$$

2. Compute $A\left(1^-,2^+,\ldots,(j-1)^+,(j+1)^+,\ldots,(n-1)^+,n^-,j^+\right)$ and show that up to a sign it equals $(1^-,2^+,3^+,\ldots,(n-1)^+,n^-)$.

   Solution.

$$
\begin{aligned}
&A\left(1^-,2^+,\ldots,(j-1)^+,(j+1)^+,\ldots,(n-1)^+,n^-,j^+\right) \\
&= \frac{\langle 1n\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle (j-1)(j+1)\rangle\ldots\langle (n-1)n\rangle\langle nj\rangle\langle j1\rangle} \\
&= \frac{s^2}{ip_{2\perp}\sqrt{\frac{-p_1^+}{\cancel{p_2^+}}}\,p_{3\perp}\sqrt{\frac{\cancel{p_2^+}}{\cancel{p_3^+}}}\ldots p_{j+1\perp}\sqrt{\frac{\cancel{p_{j-1}^+}}{\cancel{p_{j+1}^+}}}\ldots i\sqrt{-p_n^-\cancel{p_{n-1}^+}}(-i)\sqrt{-p_n^-\cancel{p_j^+}}(-i)p_{j\perp}\sqrt{\frac{-p_1^+}{\cancel{p_j^+}}}} \\
&= \frac{s}{p_{2\perp}p_{3\perp}\cdots p_{n-1\perp}}\,. \qquad \text{(H.170)}
\end{aligned}
$$

3. In eq. (1.199), it has been shown that

$$
M_n^{\text{tree}}|_{y_2\gg y_3\gg\ldots\gg y_{n-1}} = g^{n-2}\left(F^{a_2}\ldots F^{a_{n-1}}\right)_{a_1a_n} A_n^{\text{tree}}(1,2,\ldots,n-1,n)\,. \qquad \text{(H.171)}
$$

   Show that in MRK, the four MHV configurations,

$$
\begin{aligned}
&\left(1^-,2^+,3^+,\ldots,(n-1)^+,n^-\right)\,, \qquad &&\left(1^-,2^+,3^+,\ldots,(n-1)^-,n^+\right)\,, \\
&\left(1^+,2^-,3^+,\ldots,(n-1)^+,n^-\right)\,, \qquad &&\left(1^+,2^-,3^+,\ldots,(n-1)^-,n^+\right)\,. \qquad \text{(H.172)}
\end{aligned}
$$

   differ only by an overall phase, and that all the other MHV configurations are power suppressed.

   Solution. We only need the colour ordering $(1,2,\ldots,n-1,n)$.

   (a) For the MHV configuration $(1^-,2^+,3^+,\ldots,(n-1)^+,n^-)$ we have eq. (H.169).

   (b) In order to compute $A((1^-,2^+,3^+,\ldots,(n-1)^-,n^+))$ we just need to replace $\langle 1n\rangle^4$ with $\langle 1(n-1)\rangle^4$ in eq. (H.169),

$$
\langle 1n\rangle = -\sqrt{(-p_1^+)(-p_n^-)} = -\sqrt{s} \simeq -\sqrt{p_2^+p_{n-1}^-}\,, \qquad \text{(H.173)}
$$

$$
\langle 1(n-1)\rangle = ip_{n-1\perp}\sqrt{\frac{-p_1^+}{p_{n-1}^+}}\,. \qquad \text{(H.174)}
$$

Use the mass-shell condition $p^+p^- = p_\perp p_\perp^*$, then

$$\langle 1(n-1)\rangle = i\sqrt{\frac{p_{n-1\perp}}{p^*_{n-1\perp}}}\sqrt{-p_1^+p_{n-1}^-} \simeq i\sqrt{\frac{p_{n-1\perp}}{p^*_{n-1\perp}}}\sqrt{s}\,, \tag{H.175}$$

since $\sqrt{\frac{p_{n-1\perp}}{p^*_{n-1\perp}}}$ is a phase, from $\langle 1n\rangle$ to $\langle 1(n-1)\rangle$ there is only an overall phase change.

(c) For the MHV configuration, $((1^+, 2^-, 3^+, \ldots, (n-1)^+, n^+))$ we just need to replace we just need to replace $\langle 1n\rangle^4$ with $\langle 2n\rangle^4$ in eq. (H.169),

$$\langle 2n\rangle = i\sqrt{-p_n^-p_2^+} \simeq i\sqrt{s}\,. \tag{H.176}$$

Thus from $\langle 1n\rangle$ to $\langle 2n\rangle$ there is only an overall phase change.

(d) For the MHV configuration, $((1^+, 2^-, 3^+, \ldots, (n-1)^-, n^+))$ we just need to replace we just need to replace in eq. (H.169) $\langle 1n\rangle^4$ with $\langle 2(n-1)\rangle^4$

$$\langle 2(n-1)\rangle = p_{n-1\perp}\sqrt{\frac{p_2^+}{p_{n-1}^+}} = \sqrt{\frac{p_{n-1\perp}}{p^*_{n-1\perp}}}\sqrt{p_2^+p_{n-1}^-} \simeq \sqrt{\frac{p_{n-1\perp}}{p^*_{n-1\perp}}}\sqrt{s}\,. \tag{H.177}$$

So again from $\langle 1n\rangle$ to $\langle 2(n-1)\rangle$ there is only an overall phase change.

(e) Every other MHV configuration will hate in the numerator $\langle jk\rangle^4$ where $j = 1, 2, n-1, n$ and $k = 3, \ldots, n-2$ or $j, k = 3, \ldots, n-2$ and it is straightforward to see that they are all power suppressed with respect to the four we just computed.

H.19 Off-shell current $J^\mu(1^+, 2^+)$

Compute the off-shell current,

$$J^\mu(1,2) = \frac{-i}{(p_1+p_2)^2}V_3^{\mu\nu\rho}(p_1,p_2)J_\nu(1)J_\rho(2)\,, \tag{H.178}$$

with

$$V_3^{\mu\nu\rho} = \frac{i}{\sqrt{2}}\left[2g^{\mu\rho}p_2^\nu - 2g^{\mu\nu}p_1^\rho + g^{\nu\rho}(p_1-p_2)^\mu\right]\,, \tag{H.179}$$

and $J^\mu(i^\pm) = \epsilon^\mu_\pm(p_i, q_i)$. For positive-helicity gluons, take the same reference vector q and show that it can be written as

$$J^\mu(1^+, 2^+) = \frac{1}{\sqrt{2}}\frac{\langle q^-|\,\gamma^\mu(\not{p}_1 + \not{p}_2)\,|q^+\rangle}{\langle q1\rangle\langle 12\rangle\langle 2q\rangle}\,. \tag{H.180}$$

Solution. In $V_3^{\mu\nu\rho}$, the $g^{\nu\rho}$ term does not contribute, because it contracts the polarization vectors yielding $\epsilon_+(1,q)\cdot\epsilon_+(2,q)=0$. Thus

$$\begin{aligned}
J^\mu(1^+,2^+) &= -\frac{i}{(p_1+p_2)^2}\frac{i}{\sqrt{2}}(2g^{\mu\rho}p_2^\nu - 2g^{\mu\nu}p_1^\rho)\frac{\langle q^-|\,\gamma_\nu\,|1^-\rangle}{\sqrt{2}\langle q1\rangle}\frac{\langle q^-|\,\gamma_\rho\,|2^-\rangle}{\sqrt{2}\langle q2\rangle} \\
&= \frac{1}{\sqrt{2}}\frac{1}{\langle 12\rangle}\Big(\frac{\langle q^-|\,\gamma_\mu \not{p}_2\,|q^+\rangle}{\langle q1\rangle\langle 2q\rangle} + \frac{\langle q^-|\,\gamma_\mu \not{p}_1\,|q^+\rangle}{\langle q1\rangle\langle 2q\rangle}\Big) \\
&= \frac{1}{\sqrt{2}}\frac{\langle q^-|\,\gamma_\mu\left(\not{p}_1+\not{p}_2\right)|q^+\rangle}{\langle q1\rangle\langle 12\rangle\langle 2q\rangle}\,,
\end{aligned} \tag{H.181}$$

where we used

$$\not{p}_i\left|q^+\right\rangle = \left|i^-\right\rangle\langle iq\rangle \quad\Leftrightarrow\quad \left|i^-\right\rangle = \frac{\not{p}_i\,|q^+\rangle}{\langle iq\rangle}\,. \tag{H.182}$$

H.20 Off-shell current $J^\mu(1^-,2^+)$ and $J^\mu(1^-,2^+,3^+)$

Compute the off-shell currents $J^\mu(1^-,2^+)$ and $J^\mu(1^-,2^+,3^+)$ using the polarization vectors

$$\epsilon_-^\mu(p_1,p_2), \qquad \epsilon_+^\mu(p_2,p_1), \qquad \epsilon_+^\mu(p_3,p_1)\,. \tag{H.183}$$

Solution. We have

$$J^\mu(1^-,2^+) = \frac{-i}{(p_1+p_2)^2}V_3^{\mu\nu\rho}(p_1,p_2)J_\nu(1^-)J_\rho(2^+)\,, \tag{H.184}$$

where in $V_3^{\mu\nu\rho}$, the $g^{\nu\rho}$ term does not contribute, because it contracts the polarization vectors yielding $\epsilon_-(1,2)\cdot\epsilon_+(2,1)=0$. Thus

$$J^\mu(1^-,2^+) = -\frac{i}{(p_1+p_2)^2}\frac{i}{\sqrt{2}}(2g^{\mu\rho}p_2^\nu - 2g^{\mu\nu}p_1^\rho)\left(-\frac{\langle 2^+|\,\gamma_\nu\,|1^+\rangle}{\sqrt{2}[21]}\right)\frac{\langle 1^-|\,\gamma_\rho\,|2^-\rangle}{\sqrt{2}\langle 12\rangle} = 0\,. \tag{H.185}$$

We have furthermore

$$\begin{aligned}
J^\mu(1^-,2^+,3^+) = \frac{-i}{P_{1,3}^2}\Big[&V_3^{\mu\nu\rho}\left(p_1,P_{2,3}\right)J_\nu(1^-)J_\rho(2^+,3^+) + V_3^{\mu\nu\rho}\left(P_{1,2},p_3\right)\underbrace{J_\nu(1^-,2^+)}_{\to 0}J_\rho(3^+) \\
&+ V_4^{\mu\nu\rho\sigma}J_\nu(1^-)J_\rho(2^+)J_\sigma(3^+)\Big]\,,
\end{aligned} \tag{H.186}$$

where in $V_3^{\mu\nu\rho}$, the $g^{\nu\rho}$ term does not contribute, because it contracts the polarization vectors yielding $\epsilon_-(1,2)\cdot\epsilon_+(p_i,1)=0$ for $i=2,3$. Likewise, $V_4^{\mu\nu\rho\sigma}$ does not contribute, so

$$\begin{aligned}
J^\mu(1^-,2^+,3^+) &= \frac{-i}{P_{1,3}^2}\frac{i}{\sqrt{2}}(2g^{\mu\rho}P_{2,3}^\nu - 2g^{\mu\nu}p_1^\rho)\left(-\frac{\langle 1^-|\,\gamma_\nu\,|2^-\rangle}{\sqrt{2}[21]}\right)\frac{1}{\sqrt{2}}\frac{\langle 1^-|\,\gamma_\rho\left(\not{p}_2+\not{p}_3\right)|1^+\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \\
&= \frac{1}{\sqrt{2}P_{1,3}^2}\frac{\langle 1^-|\,\gamma_\mu \not{P}_{2,3}\,|1^+\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\left(-\frac{\langle 1^-|\,\not{P}_{2,3}\,|2^-\rangle}{[21]}\right)\,.
\end{aligned} \tag{H.187}$$

Note that $\not{P}_{2,3}\,|2^-\rangle = \not{p}_3\,|2^-\rangle$ and of course $\not{p}_i{}^2 = 0$, and we can write

$$-\frac{\langle 1^-|\,\not{P}_{2,3}\,|2^-\rangle}{[21]} = \frac{\langle 1^-|\,\not{p}_3\not{p}_2\,|1^+\rangle}{P_{1,2}^2} = \frac{\langle 1^-|\,\not{p}_3\not{P}_{1,3}\,|1^+\rangle}{P_{1,2}^2}\,. \tag{H.188}$$

So,

$$J^\mu(1^-,2^+,3^+) = \frac{1}{\sqrt{2}}\frac{\langle 1^-|\,\gamma_\mu\not{P}_{2,3}\,|1^+\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\frac{\langle 1^-|\,\not{p}_3\not{P}_{1,3}\,|1^+\rangle}{P_{1,2}^2P_{1,3}^2}\,, \tag{H.189}$$

which is the first non-trivial case of the formula for the $J^\mu(1^-,2^+,\ldots,n^+)$ current.

H.21 Eikonal identity

Prove the "eikonal identity"

$$\sum_{i=j}^{k-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} = \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle}\,. \tag{H.190}$$

Hint Prove it by induction using Schouten identity.

Solution. The first term in the induction, $k = j+1$, such that $i = j = k-1$ is trivially fulfilled. Let us suppose that the identity holds when $j \le k-2$,

$$\sum_{i=j}^{k-2}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} = \frac{\langle j(k-1)\rangle}{\langle jq\rangle\langle q(k-1)\rangle}\,, \tag{H.191}$$

then

$$\begin{aligned}\sum_{i=j}^{k-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} &= \frac{\langle j(k-1)\rangle}{\langle jq\rangle\langle q(k-1)\rangle} + \frac{\langle (k-1)k\rangle}{\langle (k-1)q\rangle\langle qk\rangle}\\ &= \frac{\langle j(k-1)\rangle\langle (k-1)q\rangle\langle qk\rangle + \langle (k-1)k\rangle\langle jq\rangle\langle q(k-1)\rangle}{\langle jq\rangle\langle q(k-1)\rangle\langle (k-1)q\rangle\langle qk\rangle}\,.\end{aligned} \tag{H.192}$$

Using Schouten identity,

$$\langle j(k-1)\rangle\langle qk\rangle + \langle jq\rangle\langle k(k-1)\rangle + \langle jk\rangle\langle (k-1)q\rangle = 0\,, \tag{H.193}$$

we can write

$$\begin{aligned}\sum_{i=j}^{k-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} &= \frac{\cancel{\langle (k-1)q\rangle}\langle jk\rangle\cancel{\langle q(k-1)\rangle}}{\langle jq\rangle\cancel{\langle q(k-1)\rangle}\cancel{\langle (k-1)q\rangle}\langle qk\rangle}\\ &= \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} \qquad \text{Q.E.D.}\end{aligned} \tag{H.194}$$

H.22 A useful identity

Prove the identity

$$\sum_{i=1}^{n-1} \frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} \left\langle q^-\right| \not{P}_{i+1,n} = \frac{\langle 1^-|\not{P}_{1,n}}{\langle 1q\rangle}\,, \tag{H.195}$$

with $P_{i,n} = p_i + \cdots + p_n$.

Hint Use the eikonal identity $\sum_{i=j}^{k-1} \frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} = \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle}$.

Hint Use the eikonal identity and induction.

SOLUTION I: We start with

$$\begin{aligned}
&\sum_{i=1}^{n-1} \frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} \left\langle q^-\right| \not{P}_{i+1,n} \\
&= \frac{\langle 12\rangle}{\langle 1q\rangle\langle q2\rangle} \left\langle q^-\right| (\not{p}_2 + \cdots + \not{p}_n) + \frac{\langle 23\rangle}{\langle 2q\rangle\langle q3\rangle} \left\langle q^-\right| (\not{p}_3 + \cdots + \not{p}_n) + \cdots + \frac{\langle (n-1)n\rangle}{\langle (n-1)q\rangle\langle qn\rangle} \left\langle q^-\right| \not{p}_n\,.
\end{aligned} \tag{H.196}$$

Now we collect the coefficients of the $\not{p}_i$ terms,

$$= \sum_{i=1}^{n-1} \frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} \left\langle q^-\right| \not{p}_n + \sum_{i=1}^{n-2} \frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} \left\langle q^-\right| \not{p}_{n-1} + \cdots + \frac{\langle 12\rangle}{\langle 1q\rangle\langle q2\rangle} \left\langle q^-\right| \not{p}_2\,, \tag{H.197}$$

and use the eikonal identity for each coefficient,

$$\begin{aligned}
&= \frac{\langle 1n\rangle}{\langle 1q\rangle\cancel{\langle qn\rangle}}\cancel{\langle qn\rangle} \left\langle n^+\right| + \frac{\langle 1(n-1)\rangle}{\langle 1q\rangle\cancel{\langle q(n-1)\rangle}}\cancel{\langle q(n-1)\rangle} \left\langle (n-1)^+\right| + \cdots + \frac{\langle 12\rangle}{\langle 1q\rangle\cancel{\langle q2\rangle}}\cancel{\langle q2\rangle} \left\langle 2^+\right| \\
&= \frac{\langle 1^-|\not{P}_{2,n}}{\langle 1q\rangle} \\
&= \frac{\langle 1^-|\not{P}_{1,n}}{\langle 1q\rangle}\,.
\end{aligned} \tag{H.198}$$

SOLUTION II: The first therm in the induction, $n = 2$, yields

$$\begin{aligned}
\frac{\langle 12\rangle}{\langle 1q\rangle\langle q2\rangle} \left\langle q^-\right| \not{p}_2 &= \frac{\langle 12\rangle}{\langle 1q\rangle\cancel{\langle q2\rangle}}\cancel{\langle q2\rangle} \left\langle 2^+\right| \\
&= \frac{\langle 1^-|\not{P}_{2,2}}{\langle 1q\rangle} \\
&= \frac{\langle 1^-|\not{P}_{1,2}}{\langle 1q\rangle}\,.
\end{aligned} \tag{H.199}$$

Let us suppose the identity holds for $i \leq n-2$,

$$\sum_{i=1}^{n-2} \frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle} \left\langle q^-\right| \not{P}_{i+1,n-1} = \frac{\langle 1^-|\not{P}_{1,n-1}}{\langle 1q\rangle} \tag{H.200}$$

then

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle}\left\langle q^-\right|\not{P}_{i+1,n} &= \frac{\langle 1^-|\not{P}_{1,n-1}}{\langle 1q\rangle}+\sum_{i=1}^{n-1}\frac{\langle i(i+1)\rangle}{\langle iq\rangle\langle q(i+1)\rangle}\left\langle q^-\right|\not{p}_n\\ &= \frac{\langle 1^-|\not{P}_{1,n-1}}{\langle 1q\rangle}+\frac{\langle 1n\rangle}{\langle 1q\rangle\cancel{\langle qn\rangle}}\cancel{\langle qn\rangle}\left\langle n^+\right|\\ &= \frac{\langle 1^-|\not{P}_{1,n}}{\langle 1q\rangle}\,,\end{aligned}\tag{H.201}$$

where we used the eikonal identity on the second summand.

H.23 Off-shell current $J^\mu(1^-,2^+,\ldots,n^+)$

The recursion relation for $J^\mu(1^-,2^+,\ldots,n^+)$ is

$$\begin{aligned}J^\mu(1^-,2^+,\ldots,n^+) = \frac{-i}{P_{1,n}^2}\Big[&V_3^{\mu\nu\rho}(p_1,P_{2,n})J_\nu(1^-)J_\rho(2^+,\ldots,n^+)\\ &+\sum_{i=3}^{n-1}V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})J_\nu(1^-,\ldots,i^+)J_\rho(i+1^+,\ldots,n^+)\\ &+\sum_{i=1}^{n-2}V_4^{\mu\nu\rho\sigma}J_\nu(1^-,\ldots,i^+)\sum_{j=i+1}^{n-1}J_\rho(i+1^+,\ldots,j^+)J_\sigma(j+1^+,\ldots,n^+)\Big]\,.\end{aligned}\tag{H.202}$$

Using induction, compute $J^\mu(1^-,2^+,\ldots,n^+)$ with $\epsilon_-^\mu(p_1,p_2)$ and $\epsilon_+^\mu(p_i,p_1)$ for $i=2,\ldots,n$.

Hint Show that $V_4^{\mu\nu\rho\sigma}$ and the $g^{\nu\rho}$-term in $V_3^{\mu\nu\rho}$ do not contribute, because they contract directly two current, yielding terms like $\langle 1^-|\,\gamma^\nu\,|2^-\rangle\,\langle 1^-|\,\gamma_\nu\gamma_\alpha\,|1^+\rangle$ and $\langle 1^-|\,\gamma^\nu\gamma^\alpha\,|1^+\rangle\,\langle 1^-|\,\gamma_\nu\gamma_\beta\,|1^+\rangle$ which can be Fierzed away, and reducing the recursion relation to

$$\begin{aligned}J^\mu(1^-,2^+,\ldots,n^+) = \frac{-i}{P_{1,n}^2}\Big[&V_3^{\mu\nu\rho}(p_1,P_{2,n})J_\nu(1^-)J_\rho(2^+,\ldots,n^+)\\ &+\sum_{i=3}^{n-1}V_3^{\mu\nu\rho}(P_{1,i},P_{i+1,n})J_\nu(1^-,\ldots,i^+)J_\rho(i+1^+,\ldots,n^+)\Big]\,.\end{aligned}\tag{H.203}$$

with $V_3^{\mu\nu\rho}(P,Q)=\frac{i}{\sqrt{2}}(2g^{\mu\rho}Q^\nu-2g^{\mu\nu}P^\rho)$.

Solution. We have

$$J^\mu(1^-,2^+,\dots,n^+) = \frac{-i}{P_{1,n}^2}\frac{i}{\sqrt{2}}\Big[\underbrace{(2g^{\mu\rho}P_{2,n}^\nu - 2g^{\mu\nu}p_1^\rho)\left(-\frac{\langle 1^-|\,\gamma_\nu\,|2^-\rangle}{\sqrt{2}[21]}\right)\frac{1}{\sqrt{2}}\frac{\langle 1^-|\,\gamma_\rho \not{P}_{2,n}\,|1^+\rangle}{\langle 12\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}}_{=:(1)}$$

$$+\underbrace{\sum_{i=3}^{n-1}(2g^{\mu\rho}P_{i+1,n}^\nu - 2g^{\mu\nu}P_{1,i}^\rho)\frac{1}{\sqrt{2}}\frac{\langle 1^-|\,\gamma_\rho\not{P}_{i+1,n}\,|1^+\rangle}{\langle 1(i+1)\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}}_{=:(2a)}$$

$$\times\underbrace{\frac{1}{\sqrt{2}}\frac{\langle 1^-|\,\gamma_\nu\not{P}_{2,i}\,|1^+\rangle}{\langle 12\rangle\cdots\langle 1i\rangle}\sum_{k=3}^{i}\frac{\langle 1^-|\,\not{p}_k\not{P}_{1,k}\,|1^+\rangle}{P_{1,k-1}^2P_{1,k}^2}}_{=:(2b)}\Big]\,. \tag{H.204}$$

with

$$\begin{aligned}(1) &= \frac{\langle 1^-|\,\gamma_\mu\not{P}_{2,n}\,|1^+\rangle}{\langle 12\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\left(-\frac{\langle 1^-|\,\not{P}_{2,n}\,|2^-\rangle}{[21]}\right)\\ &= \frac{\langle 1^-|\,\gamma_\mu\not{P}_{2,n}\,|1^+\rangle}{\langle 12\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\frac{\langle 1^-|\,\not{P}_{2,n}\not{p}_2\,|1^+\rangle}{P_{1,2}^2}\,,\end{aligned} \tag{H.205}$$

and

$$\begin{aligned}(2a)\times(2b) = \sum_{i=3}^{n-1}\Big(&\frac{\langle 1^-|\,\not{P}_{i+1,n}\not{P}_{2,i}\,|1^+\rangle}{\langle 12\rangle\cdots\langle i1\rangle}\frac{\langle 1^-|\,\gamma_\mu\not{P}_{i+1,n}\,|1^+\rangle}{\langle 1(i+1)\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\\ &-\frac{\langle 1^-|\,\gamma_\mu\not{P}_{2,i}\,|1^+\rangle}{\langle 12\rangle\cdots\langle i1\rangle}\frac{\langle 1^-|\,\not{P}_{1,i}\not{P}_{i+1,n}\,|1^+\rangle}{\langle 1(i+1)\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\Big)\sum_{k=3}^{i}\frac{\langle 1^-|\,\not{p}_k\not{P}_{1,k}\,|1^+\rangle}{P_{1,k-1}^2P_{1,k}^2}\,.\end{aligned} \tag{H.206}$$

The numerators can be summed,

$$\left\langle 1^-\right|\not{P}_{i+1,n}\not{P}_{2,i}\left|1^+\right\rangle\left\langle 1^-\right|\gamma_\mu\not{P}_{2,n}\left|1^+\right\rangle\,, \tag{H.207}$$

and we add

$$\left\langle 1^-\right|\not{P}_{i+1,n}\not{P}_{i+1,n}\left|1^+\right\rangle = P_{i+1,n}^2\left\langle 1^-\middle|1^+\right\rangle = 0\,, \tag{H.208}$$

such that

$$(2a)\times(2b) = \frac{\langle 1^-|\,\gamma_\mu\not{P}_{2,n}\,|1^+\rangle}{\langle 12\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\sum_{i=3}^{n-1}\frac{\langle i(i+1)\rangle}{\langle (i1)\rangle\langle 1(i+1)\rangle}\left\langle 1^-\right|\not{P}_{i+1,n}\not{P}_{2,n}\left|1^+\right\rangle\sum_{k=3}^{i}\frac{\langle 1^-|\,\not{p}_k\not{P}_{1,k}\,|1^+\rangle}{P_{i,k-1}^2P_{1,k}^2}\,. \tag{H.209}$$

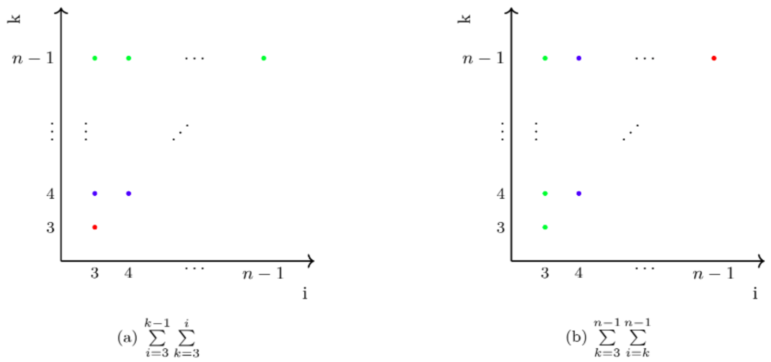


Figure H.4: Sketch of the proof of the nested sum rearrangement. Same coloured dots represent one term at a fixed i (left arrangement) or k (right arrangement) respectively.

Let us deal in particular with

$$\sum_{i=3}^{n-1} \frac{\langle i(i+1)\rangle}{\langle (i1)\rangle\langle 1(i+1)\rangle} \left\langle 1^- \right| \not{P}_{i+1,n} \sum_{k=3}^{i} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} \,. \tag{H.210}$$

Using the sum rearrangement shown in fig. H.4, we can write

$$\begin{aligned}
&\sum_{i=3}^{n-1} \frac{\langle i(i+1)\rangle}{\langle (i1)\rangle\langle 1(i+1)\rangle} \left\langle 1^- \right| \not{P}_{i+1,n} \sum_{k=3}^{i} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} \\
&\quad = \sum_{k=3}^{n-1} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} \sum_{i=k}^{n-1} \frac{\langle i(i+1)\rangle}{\langle (i1)\rangle\langle 1(i+1)\rangle} \left\langle 1^- \right| \not{P}_{i+1,n} \\
&\quad = \sum_{k=3}^{n-1} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} \frac{\langle k^-|\not{P}_{k,n}}{\langle k1\rangle} \,,
\end{aligned} \tag{H.211}$$

where we used the same derivation as for the “useful identity” in sec. H.22 to go from the second to last to the last line. So we get

$$\begin{aligned}
J^\mu(1^-,2^+,\dots,n^+) &= \frac{1}{\sqrt{2}P^2_{1,n}} \frac{\langle 1^-|\gamma_\mu \not{P}_{2,n}|1^+\rangle}{\langle 12\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle} \\
&\quad \times \Big[\frac{\langle 1^-|\not{P}_{2,n}\not{p}_2|1^+\rangle}{P^2_{1,2}} + \sum_{k=3}^{n-1} \frac{\langle k^-|\not{P}_{k+1,n}\not{P}_{2,n}|1^+\rangle}{\langle k1\rangle} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}}\Big] \,.
\end{aligned} \tag{H.212}$$

We will focus now on the term in the squared bracket and use $P_{1,n} = P_{1,k} + P_{k+1,n}$,

$$\begin{aligned}
&\sum_{k=3}^{n-1} \frac{\langle k^-|\not{P}_{k+1,n}\not{P}_{2,n}|1^+\rangle}{\langle k1\rangle} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} \\
&\quad = \sum_{k=3}^{n-1} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} \Big[(P_{1,n})^2 - \frac{\langle k^-|\not{P}_{1,k}\not{P}_{2,n}|1^+\rangle}{\langle k1\rangle}\Big] \\
&\quad = \sum_{k=3}^{n} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} (P_{1,n})^2 - \frac{\langle 1^-|\not{p}_n \not{P}_{1,n}|1^+\rangle}{P^2_{1,n-1}} - \sum_{k=3}^{n-1} \frac{\langle 1^-|\not{p}_k \not{P}_{1,k}|1^+\rangle}{P^2_{1,k-1}P^2_{1,k}} \frac{\langle k^-|\not{P}_{1,k}\not{P}_{2,n}|1^+\rangle}{\langle k1\rangle} \,,
\end{aligned} \tag{H.213}$$

where the highlighted term corresponds to the expected result. To finalise the prove we have to show, that the remaining term,

$$R = \frac{\langle 1^- |\, \slashed{P}_{2,n} \slashed{p}_2 \,|1^+\rangle}{P_{1,2}^2} - \frac{\langle 1^- |\, \slashed{p}_n \slashed{P}_{1,n} \,|1^+\rangle}{P_{1,n-1}^2} - \sum_{k=3}^{n-1} \frac{\langle 1^- |\, \slashed{p}_k \slashed{P}_{1,k} \,|1^+\rangle}{P_{1,k-1}^2 P_{1,k}^2} \frac{\langle k^- |\, \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle}{\langle k1\rangle}\,, \tag{H.214}$$

vanishes.

We start by rewriting the sum,

$$\begin{aligned}
-\sum_{k=3}^{n-1} \frac{\langle 1^- |\, \slashed{p}_k \slashed{P}_{1,k} \,|1^+\rangle}{P_{1,k-1}^2 P_{1,k}^2} \frac{\langle k^- |\, \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle}{\langle k1\rangle} &= -\sum_{k=3}^{n-1} (-\cancel{\langle k1\rangle}) \frac{\langle k^+ |\, \slashed{P}_{1,k} \,|1^+\rangle}{P_{1,k-1}^2 P_{1,k}^2} \frac{\langle k^- |\, \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle}{\cancel{\langle k1\rangle}} \\
&= \sum_{k=3}^{n-1} \frac{\langle 1^- |\, \slashed{P}_{1,k} \slashed{p}_k \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,k-1}^2 P_{1,k}^2} \\
(\text{use Clifford algebra}) &= \sum_{k=3}^{n-1} \frac{2(p_k P_{1,k-1}) \langle 1^- |\, \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle - (P_{1,k})^2 \langle 1^- |\, \slashed{p}_k \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,k-1}^2 P_{1,k}^2} \\
&= \sum_{k=3}^{n-1} \frac{(P_{1,k}^2 - P_{1,k-1}^2) \langle 1^- |\, \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle - (P_{1,k})^2 \langle 1^- |\, \slashed{p}_k \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,k-1}^2 P_{1,k}^2} \\
&= \sum_{k=3}^{n-1} \frac{\langle 1^- |\, \slashed{P}_{1,k-1} \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,k-1}^2} - \sum_{k=3}^{n-1} \frac{\langle 1^- |\, \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,k}^2} \\
(\text{extend sums} + k \to k-1 \text{ in second sum}) &= \cancel{\sum_{k=3}^{n} \frac{\langle 1^- |\, \slashed{P}_{1,k-1} \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,k-1}^2}} - \frac{\langle 1^- |\, \slashed{P}_{1,n-1} \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,n-1}^2} \\
&\quad - \cancel{\sum_{k=2}^{n-1} \frac{\langle 1^- |\, \slashed{P}_{1,k} \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,k}^2}} + \frac{\langle 1^- |\, \slashed{P}_{1,2} \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,2}^2} \\
(\text{use } P_{1,n-1} = P_{1,n} - p_n \text{ and } \langle 11\rangle = 0) &= \frac{\langle 1^- |\, \slashed{p}_n \slashed{P}_{2,n} \,|1^+\rangle}{P_{1,n-1}^2} - \frac{\langle 1^- |\, \slashed{P}_{2,n} \slashed{p}_2 \,|1^+\rangle}{P_{1,2}^2}\,.
\end{aligned} \tag{H.215}$$

This rewriting makes obvious, that the remainder R in (H.214) indeed vanishes, which is what we wanted to show.

H.24 Quark-quark scattering

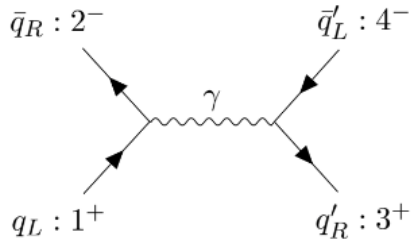


Figure H.5: $q\bar{q} \to q'\bar{q}'$ scattering for the amplitude $M_4(1_q^+, 2_{\bar{q}}^-, 3_{q'}^+, 4_{\bar{q}'}^-)$.

Let us consider quark-quark scattering with different flavours q, q' as shown in fig. H.5.

i) Compute the amplitude $M_4(1_q^+, 2_{\bar{q}}^-, 3_{q'}^+, 4_{\bar{q}'}^-)$.

ii) Square the amplitude.

iii) Compute the other helicity configurations and sum over helicities.

Hint Use the results of $e^+e^- \to \mu^+\mu^-$.

Then repeat steps i), ii) and iii) in the case where the incoming and outgoing quarks have the *same* flavour.

Hint You must subtract the contribution with quarks 1 and 3 exchanged. Why?

Solution. We have

$$M_4(1_q^+, 2_{\bar{q}}^-, 3_{q'}^+, 4_{\bar{q}'}^-) = g^2 (T^a)_{i_1}^{\bar{i}_2} (T^a)_{i_3}^{\bar{i}_4} A(1_q^+, 2_{\bar{q}}^-, 3_{q'}^+, 4_{\bar{q}'}^-)\,, \tag{H.216}$$

where up to a factor 2, the colour-stripped amplitude is the same as the amplitude for $e^+e^- \to \mu^+\mu^-$ we computed in eq. (H.30), so it is

$$iA(1_q^+, 2_{\bar{q}}^-, 3_{q'}^+, 4_{\bar{q}'}^-) = \frac{i}{s_{12}} \langle 24\rangle [31]\,. \tag{H.217}$$

The colour factor for the squared amplitude is trivially computed to be

$$\mathrm{tr}\big(T^a T^b\big)\,\mathrm{tr}\big(T^a T^b\big) = \delta^{ab}\delta^{ab} = N_c^2 - 1\,, \tag{H.218}$$

so

$$|M_4(1_q^+, 2_{\bar{q}}^-, 3_{q'}^+, 4_{\bar{q}'}^-)|^2 = g^4 (N_c^2 - 1) \frac{s_{13}^2}{s_{12}^2}\,. \tag{H.219}$$

For $|M_4(1_q^+, 2_{\bar{q}}^-, 3_{q'}^-, 4_{\bar{q}'}^+)|^2$ we can use charge conjugation on the current of the outgoing quarks $4 \leftrightarrow 3$ to immediately get

$$|M_4(1_q^+, 2_{\bar{q}}^-, 3_{q'}^-, 4_{\bar{q}'}^+)|^2 = g^4 (N_c^2 - 1) \frac{s_{14}^2}{s_{12}^2}\,. \tag{H.220}$$

The last two configurations are obtained from the already computed one by parity, so the square does not change and we have

$$\sum_{\text{hel}} |M_4(1_q, 2_{\bar{q}}, 3_{q'}, 4_{\bar{q}'})|^2 = 2g^4 (N_c^2 - 1) \frac{s_{13}^2 + s_{14}^2}{s_{12}^2}\,, \tag{H.221}$$

or if we average over the initial colours and helicities, i.e. divide by $4N_c^2$,

$$\sum_{\text{hel}} |\overline{M}_4(1_q, 2_{\bar{q}}, 3_{q'}, 4_{\bar{q}'})|^2 = \frac{g^4 (N_c^2 - 1)}{2N_c^2} \frac{s_{13}^2 + s_{14}^2}{s_{12}^2}\,. \tag{H.222}$$

In the case where the incoming and outgoing quarks have the same flavour,

$$
\begin{aligned}
M_4(1_q^+,2_{\bar{q}}^-,3_q^+,4_{\bar{q}}^-) &= g^2(T^a)_{i_1}^{\bar{i}_2}(T^a)_{i_3}^{\bar{i}_4}A(1_q^+,2_{\bar{q}}^-,3_{q'}^+,4_{\bar{q}'}^-) - g^2(T^a)_{i_3}^{\bar{i}_2}(T^a)_{i_1}^{\bar{i}_4}A(3_q^+,2_{\bar{q}}^-,1_q^+,4_{\bar{q}}^-) \\
&= g^2\Big[(T^a)_{i_1}^{\bar{i}_2}(T^a)_{i_3}^{\bar{i}_4}\underbrace{\frac{\langle 24\rangle[31]}{s_{12}}}_{=:a_1} - (T^a)_{i_3}^{\bar{i}_2}(T^a)_{i_1}^{\bar{i}_4}\underbrace{\frac{\langle 24\rangle[13]}{s_{32}}}_{=:a_2}\Big]\,. \qquad \text{(H.223)}
\end{aligned}
$$

To square the amplitude we need the additional the colour factor,

$$
\operatorname{tr}\Big(T^aT^bT^aT^b\Big) = -\frac{N_c^2-1}{N_c}\,, \qquad \text{(H.224)}
$$

and the additional term,

$$
a_1a_2^* = \frac{\langle 24\rangle[31][42]\langle 31\rangle}{s_{12}s_{14}}\,, \qquad \text{(H.225)}
$$

so

$$
|M_4(1_q^+,2_{\bar{q}}^-,3_q^+,4_{\bar{q}}^-)|^2 = g^4\Big[\left(\frac{s_{13}^2}{s_{12}^2}+\frac{s_{13}^2}{s_{14}^2}\right)(N_c^2-1) - 2\frac{s_{13}^2}{s_{12}s_{14}}\frac{N_c^2-1}{N_c}\Big]\,. \qquad \text{(H.226)}
$$

The other helicity configurations are

$$
M_4(1_q^+,2_{\bar{q}}^-,3_q^-,4_{\bar{q}}^+) = g^2(T^a)_{i_1}^{i_2}(T^a)_{i_3}^{i_4}\frac{\langle 23\rangle[41]}{s_{12}}\,, \qquad \text{(H.227)}
$$

and the one with the quarks 1 and 3 interchanged,

$$
M_4(3_q^+,2_{\bar{q}}^-,1_q^-,4_{\bar{q}}^+) = g^2(T^a)_{i_3}^{i_2}(T^a)_{i_1}^{i_4}\frac{\langle 21\rangle[43]}{s_{32}}. \qquad \text{(H.228)}
$$

Note that they cannot interfere. With

$$
|M_4(1_q^+,2_{\bar{q}}^-,3_q^-,4_{\bar{q}}^+)|^2 = g^4(N_c^2-1)\frac{s_{14}^2}{s_{12}^2}\,, \qquad \text{(H.229)}
$$

$$
|M_4(3_q^+,2_{\bar{q}}^-,1_q^-,4_{\bar{q}}^+)|^2 = g^4(N_c^2-1)\frac{s_{34}^2}{s_{32}^2}\,, \qquad \text{(H.230)}
$$

and the other helicity configurations obtained by parity we have

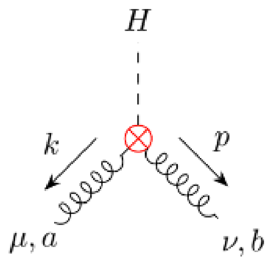
$$
\begin{aligned}
\sum_{\text{hel}}|\overline{M}_4(1_q,2_{\bar{q}},3_q,4_{\bar{q}})|^2 &= \frac{1}{4N_c^2}\sum_{\text{hel}}|M_4(1_q,2_{\bar{q}},3_q,4_{\bar{q}})|^2 \\
&= \frac{g^4(N_c^2-1)}{2N_c^2}\Big[\frac{s_{13}^2+s_{14}^2}{s_{12}^2}+\frac{s_{13}^2+s_{12}^2}{s_{14}^2}-\frac{2}{N_c}\frac{s_{13}^2}{s_{12}s_{14}}\Big]\,. \qquad \text{(H.231)}
\end{aligned}
$$

H.25 $gg \to H$ and $gg \to Hg$ scattering in HEFT

1. Compute the amplitude for Higgs production from gluon fusion, $gg \to H$, in the Higgs Effective field theory (HEFT), for the two independent helicity configurations 1^+2^+ and 1^-2^+.

   Hint The color structure is the same as for the gluon amplitudes, thus the same color decomposition of the amplitude and the same properties of the sub-amplitudes (cyclicity, reflection, $U(1)$ decoupling) hold.

   Hint In HEFT, the color-ordered Feynman rules are



$$\to \left(-\frac{\alpha_s}{3\pi v}\right) i\,\mathrm{tr}\left(T^a T^b\right)\left((p\cdot k)g^{\mu\nu} - p^\mu k^\nu\right) \tag{1}$$

   where v is the Higgs vacuum expectation value.

   Hint Note that it is not possible to choose the same reference null vector k^μ for all the polarization vectors.

   Solution The $gg \to H$ amplitude is

$$M_2 = -\frac{\alpha_s}{3\pi v}\,\mathrm{tr}(T^{a_1}T^{a_2})A_2(1,2)\,. \tag{H.232}$$

   There is only one sub-amplitude,

$$\begin{aligned} iA_2(1,2) &= i((p_1p_2)g^{\mu_1\mu_2} - p_1^{\mu_2}p_2^{\mu_1})\epsilon^{\mu_1}(p_1)\epsilon^{\mu_2}(p_2) \\ &= i((p_1p_2)(\epsilon(p_1)\epsilon(p_2)) - (p_1\epsilon(p_2))(p_2\epsilon(p_1)))\,. \end{aligned} \tag{H.233}$$

   Since we cannot find a common reference null vector k^μ, an obvious choice is to take $\epsilon^\mu_{(1,2)}$ and $\epsilon^\mu_{(2,1)}$ as the polarization vectors, where we used the notation $\epsilon^\mu(p_i,p_j) = \epsilon^\mu_{(i,j)}$. Since

$$p_i\epsilon^\pm_{(i,j)} = p_i\epsilon^\pm_{(j,i)} = 0\,, \tag{H.234}$$

   the second term in A_2 does not contribute. So

$$iA_2(1,2) = i\frac{m_H^2}{2}(\epsilon_{(1,2)}\epsilon_{(2,1)})\,, \tag{H.235}$$

where we used $p_1 + p_2 + p_H = 0$ and $2p_1p_2 = m_H^2$. We have for 1^+2^+,

$$\epsilon^{+\mu}_{(j,k)} = \frac{\langle k^-|\,\gamma^\mu\,|j^-\rangle}{\sqrt{2}\langle kj\rangle}\,, \tag{H.236}$$

so

$$\begin{aligned} iA_2(1^+,2^+) &= i\frac{m_H^2}{2}\frac{\langle 2^-|\,\gamma^\mu\,|1^-\rangle}{\sqrt{2}\langle 21\rangle}\frac{\langle 1^-|\,\gamma_\mu\,|2^-\rangle}{\sqrt{2}\langle 12\rangle} \\ &= i\frac{m_H^2}{2}\frac{2\langle 12\rangle[21]}{2\langle 12\rangle^2} \\ &= i\frac{m_H^2}{2}\frac{[21]}{\langle 12\rangle}\,. \end{aligned} \tag{H.237}$$

The $(--)$-configuration is obtained by parity.

For the $(+-)$-configuration we have $\epsilon^+_{(1,2)}\epsilon^-_{(2,1)} = 0$, so neither $(+-)$ nor $(-+)$ contribute.

2. On physical grounds, can you show that the result for 1^+2^- holds to all loops?

   Solution The Higgs boson has no spin, and so no angular momentum. In the Higgs rest frame, the two gluons are back-to-back, they fly in opposite direction. For their total angular momentum to vanish, they must have equal helicities. For the same reason, $q\bar{q} \to H$ cannot exist for massless quarks, since the helicity is conserved on the quark line. It exist only for massive quarks wit a helicity flip on the quark line.

   So $A(1^-,2^+) = 0$ to all loops.

3. Compute the squared amplitude for $gg \to H$.

   Solution We have

$$\begin{aligned} M_2(+,+) &= \mathrm{tr}(T^{a_1}T^{a_2})\frac{\alpha_s}{3\pi v}\frac{m_H^2}{2}\frac{[12]}{\langle 12\rangle} \\ &= \delta^{a_1a_2}\frac{\alpha_s}{3\pi v}\frac{m_H^2}{2}\frac{[12]}{\langle 12\rangle}\,, \end{aligned} \tag{H.238}$$

   with

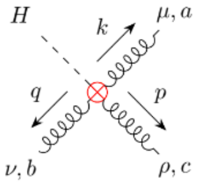
$$|M_2(+,+)|^2 = (\frac{\alpha_s}{3\pi v})^2(N_c^2-1)\frac{m_H^2}{4}\,. \tag{H.239}$$

   With $|M_2(+,+)|^2 = |M_2(-,-)|^2$ the helicity and colour-averaged matrix element is

$$\begin{aligned} \overline{|M_2(+,+)|}^2 &= \frac{1}{2(N_c^2-1)}\frac{1}{2(N_c^2-1)}\Big(2(\frac{\alpha_s}{3\pi v})^2(N_c^2-1)\frac{m_H^2}{4}\Big) \\ &= (\frac{\alpha_s}{3\pi v})^2\frac{1}{8(N_c^2-1)}m_H^4\,. \end{aligned} \tag{H.240}$$

4. In HEFT, compute the $gg \to Hg$ amplitude for the helicity configurations $1^+2^+3^+$ and $1^-2^+3^+$.

Hint The relevant vertex is:



$$\to \left(-\frac{\alpha_s}{3\pi v}\right)\frac{ig_s}{\sqrt{2}}\,\mathrm{tr}\left(T^aT^bT^c\right)\left[g^{\mu\rho}(q-k)^\nu + g^{\mu\nu}(k-p)^\rho + g^{\rho\nu}(p-q)^\mu\right] + \textit{non-cycl. perm.} \tag{2}$$

.

Solution The mutiperipheral colour decomposition of the $gg \to Hg$ amplitude is

$$M(1,2,3) = -\frac{\alpha_s}{3\pi v} g F^{abc} A_3(1,2,3) \qquad \text{with} \qquad F^{abc} = i\sqrt{2} f^{abc}\,. \tag{H.241}$$

There is only one sub-amplitude $A_3(1,2,3)$ to compute (in the trace-based decomposition there are two sub-amplitudes to compute, $A_3(1,2,3)$ and $A_3(1,3,2)$, but they are related by reflection and cyclicity). As polarization vectors we choose (with the same shorthand notation as above)

$$\epsilon(1,2), \qquad \epsilon(2,1) \qquad \epsilon(3,1)\,, \tag{H.242}$$

where again $\epsilon(i,j)p_i = \epsilon(i,j)p_j = 0$. For $(-++)$, we have further

$$\epsilon^+_{(2,1)}\epsilon^-_{(1,2)} = \epsilon^+_{(3,1)}\epsilon^-_{(1,2)} = \epsilon^+_{(2,1)}\epsilon^+_{(3,1)} = 0\,. \tag{H.243}$$

Let us compute the Feynman diagrams using the colour-ordered Feynman rules. From the ggH-vertex, we see immediately that

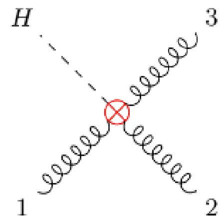


Figure H.6: A_a diagram.

$$\begin{aligned} iA_a(-++) =& \frac{i}{\sqrt{2}}\left[(p_1-p_2)_{\mu_3}\cancelto{0}{g_{\mu_1\mu_2}} + (p_2-p_3)_{\mu_1}\cancelto{0}{g_{\mu_2\mu_3}} + (p_3-p_1)_{\mu_2}\cancelto{0}{g_{\mu_1\mu_3}}\right]\epsilon^{-\mu_1}_{(1,2)}\epsilon^{+\mu_2}_{(2,1)}\epsilon^{+\mu_3}_{(3,1)} \\ =& 0\,. \end{aligned} \tag{H.244}$$

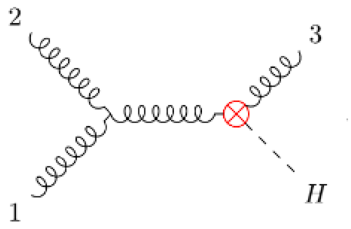


Figure H.7: A_b diagram.

$$
\begin{aligned}
iA_b(-++) =& i\Big(p_3 \cdot (p_1+p_2) g^{\nu}_{\mu_3} - p_3^{\nu}(p_1+p_2)_{\mu_3}\Big)\frac{-i}{s_{12}} \\
&\times \frac{i}{\sqrt{2}}\Big(\cancelto{0}{g_{\mu_1\mu_2}}(p_1-p_2)_{\nu} + g_{\mu_2\nu}2\cancelto{0}{p_{2_{\mu_1}}} - g_{\mu_1\nu}2\cancelto{0}{p_{1_{\mu_2}}}\Big)\epsilon^{-\mu_1}_{(1,2)}\epsilon^{+\mu_2}_{(2,1)}\epsilon^{+\mu_3}_{(3,1)} \\
=& 0\,.
\end{aligned}
\tag{H.245}
$$

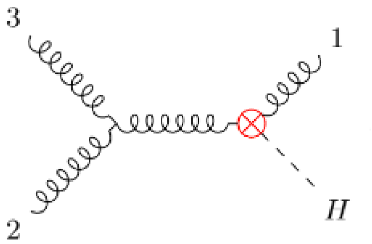


Figure H.8: A_c diagram.

$$
\begin{aligned}
A_c(-++) =& i\Big(p_1 \cdot (p_2+p_3) \cancelto{0}{g^{\nu}_{\mu_1}} - \cancelto{0}{p_1^{\nu}}(p_2+p_3)_{\mu_1}\Big)\frac{-i}{s_{23}} \\
&\times \frac{i}{\sqrt{2}}\Big(\cancelto{0}{g_{\mu_2\mu_3}}(p_2-p_3)_{\nu} + g_{\mu_3\nu}2p_{3_{\mu_2}} - g_{\mu_2\nu}2p_{2_{\mu_3}}\Big)\epsilon^{-\mu_1}_{(1,2)}\epsilon^{+\mu_2}_{(2,1)}\epsilon^{+\mu_3}_{(3,1)} \\
=& 0\,.
\end{aligned}
\tag{H.246}
$$

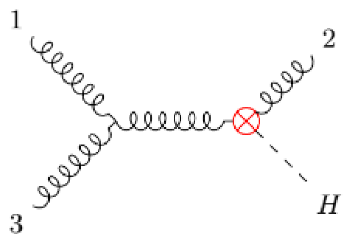


Figure H.9: A_d diagram.

$$
\begin{aligned}
iA_d(-++) =& i\Big(p_2\cdot(p_1+p_3)\cancelto{0}{g_{\mu_2}} - p_2^\nu(\cancelto{0}{p_1}+p_3)_{\mu_2}\Big)\frac{-i}{s_{13}} \\
&\times\frac{i}{\sqrt{2}}\Big(\cancelto{0}{g_{\mu_3\mu_1}}(p_3-p_1)_\nu + g_{\mu_1\nu}2\cancelto{0}{p_{1\mu_3}} - g_{\mu_3\nu}2p_{3\mu_1}\Big)\epsilon^{-\mu_1}_{(1,2)}\epsilon^{+\mu_2}_{(2,1)}\epsilon^{+\mu_3}_{(3,1)} \\
=&\frac{i}{\sqrt{2}}\frac{2}{s_{13}}(-\epsilon^+_{(2,1)}p_3)(-p_2\epsilon^+_{(3,1)})(p_3\epsilon^-_{(1,2)}) \\
=&\frac{i\sqrt{2}}{s_{13}}\frac{\langle 1^-|\not{3}|2^-\rangle}{\sqrt{2}\langle 12\rangle}\frac{\langle 1^-|\not{2}|3^-\rangle}{\sqrt{2}\langle 13\rangle}\Big(-\frac{\langle 2^+|\not{3}|1^+\rangle}{\sqrt{2}[21]}\Big) \\
=&-\frac{i}{2[13]\cancel{\langle 31\rangle}}\frac{\cancel{\langle 13\rangle}[32]\cancel{\langle 12\rangle}[23][23]\cancel{\langle 31\rangle}}{\cancel{\langle 12\rangle}\cancel{\langle 13\rangle}[21]} \\
=&\frac{i}{2}\frac{[23]^3}{[12][31]}\,.
\end{aligned}
\tag{H.247}
$$

So the sub-amplitude is

$$
iA_3(-++) = \frac{i}{2}\frac{[23]^4}{[12][23][31]}\,, \tag{H.248}
$$

i.e. it is a $\overline{\text{MHV}}$-amplitude (note that it exists with three gluons and real momenta because this is actually four-pt kinematics $p_1+p_2+p_3=-p_H$).

The case $(+++)$ is more computationally involved, since we have $\epsilon^+_{(2,1)}\epsilon^+_{(1,2)}\neq 0$ and $\epsilon^+_{(3,1)}\epsilon^+_{(1,2)}\neq 0$. We will furthermore use the A_a, A_b, A_c and A_d notation of the diagrams shown above.

We have

$$
\begin{aligned}
iA_a(+++) =& \frac{i}{\sqrt{2}}\Big[(\cancelto{0}{p_1} - p_2)_{\mu_3} g_{\mu_1\mu_2} + (p_2 - p_3)_{\mu_1} \cancelto{0}{g_{\mu_2\mu_3}} + (p_3 - \cancelto{0}{p_1})_{\mu_2} g_{\mu_1\mu_3}\Big] \epsilon^{+\mu_1}_{(1,2)} \epsilon^{+\mu_2}_{(2,1)} \epsilon^{+\mu_3}_{(3,1)} \\
=& \frac{i}{\sqrt{2}}\Big((-p_2\epsilon^+_{(31)})(\epsilon^+_{(12)}\epsilon^+_{(21)}) + (p_3\epsilon^+_{(21)})(\epsilon^+_{(12)}\epsilon^+_{(31)})\Big)\,,
\end{aligned} \tag{H.249}
$$

$$
\begin{aligned}
iA_b(+++) =& i\Big(p_3\cdot(p_1+p_2)g^{\nu}_{\mu_3} - p_3^{\nu}(\cancelto{0}{p_1} + p_2)_{\mu_3}\Big)\frac{-i}{s_{12}} \\
&\times \frac{i}{\sqrt{2}}\Big(g_{\mu_1\mu_2}(p_1 - p_2)_{\nu} + g_{\mu_2\nu}2\cancelto{0}{p_{2\mu_1}} - g_{\mu_1\nu}2\cancelto{0}{p_{1\mu_2}}\Big)\epsilon^{+\mu_1}_{(1,2)} \epsilon^{+\mu_2}_{(2,1)} \epsilon^{+\mu_3}_{(3,1)} \\
=& \frac{i}{\sqrt{2}}\frac{1}{s_{12}}\Big((p_3\cdot(p_1+p_2))(-p_2\epsilon^+_{(3,1)}) - (p_3\cdot(p_1-p_2))(p_2\epsilon^+_{(3,1)})\Big)(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)}) \\
=& -\frac{i}{\sqrt{2}}\frac{2}{s_{12}}(p_1p_3)(p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)})\,,
\end{aligned} \tag{H.250}
$$

$$
\begin{aligned}
iA_c(+++) =& i\Big(p_1\cdot(p_2+p_3)g^{\nu}_{\mu_1} - \cancelto{0}{p_1^{\nu}}(p_2+p_3)_{\mu_1}\Big)\frac{-i}{s_{23}} \\
&\times \frac{i}{\sqrt{2}}\Big(\cancelto{0}{g_{\mu_2\mu_3}}(p_2 - p_3)_{\nu} + g_{\mu_3\nu}2p_{3\mu_2} - g_{\mu_2\nu}2p_{2\mu_3}\Big)\epsilon^{+\mu_1}_{(1,2)} \epsilon^{+\mu_2}_{(2,1)} \epsilon^{+\mu_3}_{(3,1)} \\
=& \frac{i}{\sqrt{2}}\frac{2}{s_{23}}(p_1\cdot(p_2+p_3))\Big((p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) - (p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)})\Big)\,,
\end{aligned} \tag{H.251}
$$

$$
\begin{aligned}
iA_d(+++) =& i\Big(p_2\cdot(p_1+p_3)g^{\nu}_{\mu_2} - p_2^{\nu}(\cancelto{0}{p_1} + p_3)_{\mu_2}\Big)\frac{-i}{s_{13}} \\
&\times \frac{i}{\sqrt{2}}\Big(g_{\mu_3\mu_1}(p_3 - p_{1\nu}) + g_{\mu_1\nu}2\cancelto{0}{p_{1\mu_3}} - g_{\mu_3\nu}2p_{3\mu_1}\Big)\epsilon^{+\mu_1}_{(1,2)} \epsilon^{+\mu_2}_{(2,1)} \epsilon^{+\mu_3}_{(3,1)} \\
=& \frac{i}{\sqrt{2}}\frac{1}{s_{13}}\Big[((p_2\cdot(p_1+p_3))((p_3 - \cancel{p_1})\cdot\epsilon^+_{(2,1)}) - ((p_3-p_1)\cdot p_2)(p_3\epsilon^+_{(2,1)}))(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) \\
&\qquad + 2(p_2\epsilon^+_{(3,1)})(p_3\epsilon^+_{(1,2)})(p_3\epsilon^+_{(2,1)})\Big] \\
=& \frac{i}{\sqrt{2}}\frac{2}{s_{13}}\Big((p_1p_2)(p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) + (p_2\epsilon^+_{(3,1)})(p_3\epsilon^+_{(1,2)})(p_3\epsilon^+_{(2,1)})\Big)\,.
\end{aligned} \tag{H.252}
$$

We collect the four contributions,

$$\begin{aligned}
iA_3(+++) =& \frac{i}{\sqrt{2}}\Big[(-p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)}) + (p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) - \frac{s_{13}}{s_{12}}(p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)}) \\
&+ \frac{s_{12}+s_{13}}{s_{23}}((p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) - (p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)})) \\
&+ \frac{s_{12}}{s_{13}}(p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) + \frac{2}{s_{13}}(p_2\epsilon^+_{(3,1)})(p_3\epsilon^+_{(1,2)})(p_3\epsilon^+_{(2,1)})\Big] \\
=& \frac{i}{\sqrt{2}}\Big[-\left(1+\frac{s_{13}}{s_{12}}+\frac{s_{12}+s_{13}}{s_{23}}\right)(p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)}) \\
&+\left(1+\frac{s_{12}+s_{13}}{s_{23}}+\frac{s_{12}}{s_{13}}\right)(p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) + \frac{2}{s_{13}}(p_2\epsilon^+_{(3,1)})(p_3\epsilon^+_{(1,2)})(p_3\epsilon^+_{(2,1)})\Big] \\
=& \frac{i}{\sqrt{2}}\Big[-\frac{(s_{12}+s_{13})(s_{12}+s_{23})}{s_{12}s_{23}}(p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)}) \\
&+\frac{(s_{12}+s_{13})(s_{13}+s_{23})}{s_{13}s_{23}}(p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) + \frac{2}{s_{13}}(p_2\epsilon^+_{(3,1)})(p_3\epsilon^+_{(1,2)})(p_3\epsilon^+_{(2,1)})\Big] .
\end{aligned} \tag{H.253}$$

In order to write them in bracket notation we use

$$(p_2\epsilon^+_{(3,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(2,1)}) = \frac{\langle 12\rangle[23]}{\sqrt{2}\langle 13\rangle}\frac{\cancel{\langle 12\rangle}[12]}{\langle 21\rangle\cancel{\langle 12\rangle}} = -\frac{[12][23]}{\sqrt{2}\langle 13\rangle}, \tag{H.254}$$

$$(p_3\epsilon^+_{(2,1)})(\epsilon^+_{(1,2)}\epsilon^+_{(3,1)}) = \frac{\cancel{\langle 13\rangle}[32]}{\sqrt{2}\langle 12\rangle}\frac{\cancel{\langle 21\rangle}[31]}{\cancel{\langle 21\rangle}\cancel{\langle 13\rangle}} = \frac{[13][23]}{\sqrt{2}\langle 12\rangle}, \tag{H.255}$$

$$(p_2\epsilon^+_{(3,1)})(p_3\epsilon^+_{(1,2)})(p_3\epsilon^+_{(2,1)}) = \frac{\cancel{\langle 12\rangle}[23]}{\sqrt{2}\cancel{\langle 13\rangle}}\frac{\langle 23\rangle[31]}{\sqrt{2}\langle 21\rangle}\frac{\cancel{\langle 13\rangle}[32]}{\sqrt{2}\langle 12\rangle} = \frac{s_{23}}{2\sqrt{2}}\frac{[13][23]}{\langle 12\rangle}, \tag{H.256}$$

and the partial amplitude is

$$\begin{aligned}
&iA_3(+++) \\
&= \frac{i}{\sqrt{2}}\Big[\frac{(s_{12}+s_{13})(s_{12}+s_{23})}{s_{12}s_{23}}\frac{[12][23]}{\sqrt{2}\langle 13\rangle} + \frac{(s_{12}+s_{13})(s_{13}+s_{23})}{s_{13}s_{23}}\frac{[13][23]}{\sqrt{2}\langle 12\rangle} + \frac{2}{s_{13}}\frac{s_{23}}{2\sqrt{2}}\frac{[13][23]}{\langle 12\rangle}\Big] \\
&= \frac{i}{2}\frac{(s_{12}+s_{13})(s_{12}+s_{23}) + (s_{12}+s_{13})(s_{13}+s_{23}) + s_{23}^2}{\langle 12\rangle\langle 23\rangle\langle 13\rangle} \\
&= \frac{i}{2}\frac{(s_{12}+s_{13}+s_{23})^2}{\langle 12\rangle\langle 23\rangle\langle 13\rangle} \\
&= -\frac{i}{2}\frac{m_H^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} .
\end{aligned} \tag{H.257}$$

where we used $(s_{12}+s_{13}+s_{23}) = (p_1+p_2+p_3)^2 = m_H^2$.

To summarize, the amplitudes are

$$iM_3(-++) = -\frac{\alpha_s}{3\pi v}gF^{a_1a_2a_3}\frac{i}{2}\frac{[23]^4}{[12][23][31]}, \tag{H.258}$$

$$iM_3(+++) = -\frac{\alpha_s}{3\pi v}gF^{a_1a_2a_3}(-\frac{i}{2})\frac{m_H^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} . \tag{H.259}$$

with $F^{abc} = i\sqrt{2} f^{abc}$.

5. Compute the squared amplitude for $gg \to Hg$.

Solution. In order to square the amplitude we use

$$f^{abc} f^{abd} = N_c \delta^{cd} \qquad \Rightarrow \qquad F^{abc} F^{cba} = 2N_c(N_c^2 - 1)\,, \tag{H.260}$$

to obtain

$$\begin{aligned} |M_3(-++)|^2 &= \Big(\frac{\alpha_s}{3\pi v} g\Big)^2 \frac{N_c(N_c^2-1)}{2} \frac{s_{23}^4}{s_{12}s_{23}s_{13}} \\ &= \frac{2\alpha_s^3}{9\pi v^2} N_c(N_c^2-1) \frac{s_{23}^4}{s_{12}s_{23}s_{13}}\,, \end{aligned} \tag{H.261}$$

$$|M_3(+++)|^2 = \frac{2\alpha_s^3}{9\pi v^2} N_c(N_c^2-1) \frac{m_H^8}{s_{12}s_{23}s_{13}}\,. \tag{H.262}$$

$|M_3(+-+)|^2$ and $|M_3(++-)|^2$ are obtained by permuting the labels $1, 2, 3$. And by parity we have

$$|M_3(---)|^2 = |M_3(+++)|^2 \qquad |M_3(+--)|^2 = |M_3(-++)|^2\,. \tag{H.263}$$

So finally, the squared amplitude, averaged over the initial state helicities and colour, is

$$\begin{aligned} \overline{|M_3(1,2,3)|}^2 &= \frac{1}{2(N_c^2-1)} \frac{1}{2(N_c^2-1)} \frac{4\alpha_s^3}{9\pi v^2} N_c(N_c^2-1) \frac{m_H^8 + s_{12}^4 + s_{13}^4 + s_{23}^4}{s_{12}s_{23}s_{13}} \\ &= \frac{\alpha_s^3}{9\pi v^2} \frac{N_c}{(N_c^2-1)} \frac{m_H^8 + s_{12}^4 + s_{13}^4 + s_{23}^4}{s_{12}s_{23}s_{13}}\,. \end{aligned} \tag{H.264}$$

6. Show how the amplitude factorizes in the soft limit, $p_1 \to 0$, and in the collinear limit, $(p_1 \cdot p_3) = 0$, and compute the eikonal factors and the splitting amplitudes.

Solution. For the soft limit $p_1 \to 0$, we use the amplitudes,

$$A_3(1^+2^+3^+) = -\frac{1}{2} \frac{m_H^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,, \tag{H.265}$$

$$A_2(2^+3^+) = -\frac{m_H^2}{2} \frac{[23]}{\langle 23\rangle}\,. \tag{H.266}$$

Note that we can write

$$A_3(1^+2^+3^+) = \frac{m_H^4}{2\langle 23\rangle^2} S(3^+, 1^+, 2^+)\,, \tag{H.267}$$

with

$$S(3^+, 1^+, 2^+) = \frac{\langle 32\rangle}{\langle 31\rangle\langle 12\rangle}\,. \tag{H.268}$$

As $p_1 \to 0$,

$$(p_2 + p_3) = -p_H \Rightarrow m_H^2 = s_{23} = \langle 23\rangle[32]\,, \tag{H.269}$$

so

$$\begin{aligned}\lim_{p_1\to 0} A_3(1^+2^+3^+) &= \frac{m_H^2\langle 23\rangle[32]}{2\langle 23\rangle^2} S(3^+,1^+,2^+) \\ &= A_2(2^+3^+) S(3^+,1^+,2^+)\,,\end{aligned} \tag{H.270}$$

which is the expected factorisation of the soft gluon 1. By parity, we obtain the amplitudes,

$$A_3(1^+2^-3^-) = -\frac{\langle 23\rangle^4}{2\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,, \tag{H.271}$$

$$A_2(2^-3^-) = -\frac{m_H^2}{2}\frac{\langle 23\rangle}{[23]}\,, \tag{H.272}$$

and we can write

$$A_3(1^+2^-3^-) = \frac{\langle 23\rangle^2}{2} S(3^-,1^+,2^-)\,, \tag{H.273}$$

with $S(3^-,1^+,2^-) = S(3^+,1^+,2^+)$ and

$$\begin{aligned}\lim_{p_1\to 0} A_3(1^+2^-3^-) &= -\frac{m_H^2}{2}\frac{\langle 23\rangle}{[23]} S(3^-,1^+,2^-) \\ &= A_2(2^-3^-) S(3^-,1^+,2^-)\,,\end{aligned} \tag{H.274}$$

i. e. the eikonal factor $S(3,1^+,2)$ does not depend on the helicities of gluon 2 and 3.
In fact, it coincides with the eikonal factor of a positive-helicity soft gluon out of a quark line, we computed in the soft limit of $e^+e^- \to q\bar{q}g$. As we anticipated there, the eikonal factor does not depend on the spin or on the parton flavour of the emitters (in this case 2 and 3).
Using

$$A_3(1^-2^+3^+) = \frac{i}{2}\frac{[23]^4}{[12][23][31]}\,, \tag{H.275}$$

and $A(2^+3^+)$ we get easily that

$$S(3,1^-,2) = -\frac{[32]}{[31][12]}\,, \tag{H.276}$$

in agreement with the eikonal factor of a negative-helicity soft gluon out of a quark line.

In the collinear limit $p_1 p_3 \to 0$, we parametrise,

$$p_3 = zP \qquad\qquad p_1 = (1-z)P\,, \tag{H.277}$$

then $p_1 + p_2 + p_3 = p_2 + P = -p_H \Rightarrow m_H^2 = s_{2P}$.

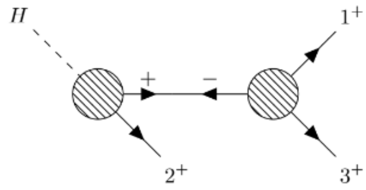


Figure H.10: $\lim_{p_1 p_3 \to 0} A(1^+ 2^+ 3^+)$.

$$\begin{aligned}
\lim_{p_1 p_3 \to 0} A(1^+ 2^+ 3^+) &= -\frac{1}{2} \frac{m_H^4}{\sqrt{z(1-z)} \langle P2 \rangle \langle 2P \rangle \langle 31 \rangle} \\
&= \frac{1}{\sqrt{z(1-z)} \langle 31 \rangle} \frac{m_H^4}{2 \langle 2P \rangle^2} \\
&= \frac{1}{\sqrt{z(1-z)} \langle 31 \rangle} \frac{m_H^2 [P2]}{2 \langle 2P \rangle} \\
&= \text{Split}_-(3^+ 1^+) A_2(2^+ P^+) \,, \qquad \text{(H.278)}
\end{aligned}$$

with

$$\text{Split}_-(3^+ 1^+) = \frac{1}{\sqrt{z(1-z)} \langle 31 \rangle} \,. \qquad \text{(H.279)}$$

The second case to consider is

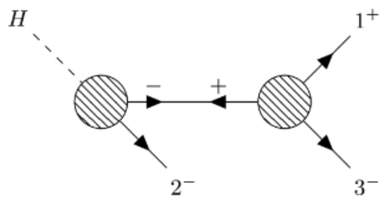


Figure H.11: $\lim_{p_1 p_3 \to 0} A(1^+ 2^- 3^-)$.

$$\begin{aligned}
\lim_{p_1 p_3 \to 0} A(1^+ 2^- 3^-) &= \lim_{p_1 p_3 \to 0} \Big(-\frac{\langle 23 \rangle^4}{2 \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \Big) \\
&= -\frac{z^2}{\sqrt{z(1-z)} \langle 31 \rangle} \frac{\langle 2P \rangle^3}{2 \langle P2 \rangle} \\
&= \frac{z^2}{\sqrt{z(1-z)} \langle 31 \rangle} \frac{\langle 2P \rangle^2}{2} \\
&= \text{Split}_+(3^- 1^+) A_2(2^- P^-) \,, \qquad \text{(H.280)}
\end{aligned}$$

with

$$\mathrm{Split}_+(3^-1^+) = \frac{z^2}{\sqrt{z(1-z)}\langle 31\rangle}\,, \tag{H.281}$$

and

$$A_2(2^-P^-) = -\frac{m_H^2}{2}\frac{\langle 2P\rangle}{[2P]} = \frac{\langle 2P\rangle^2}{2}\,. \tag{H.282}$$

The third case is

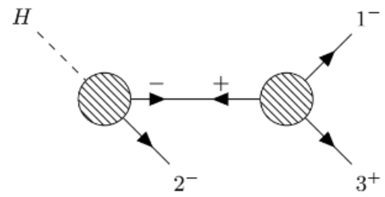


Figure H.12: $\lim_{p_1p_3\to 0} A(1^-2^-3^+)$.

$$\begin{aligned}\lim_{p_1p_3\to 0} A(1^-2^-3^+) &= -\frac{(1-z)^2}{\sqrt{z(1-z)}\langle 31\rangle}\frac{\langle P2\rangle^3}{2\langle 2P\rangle} \\ &= \frac{(1-z)^2}{\sqrt{z(1-z)}\langle 31\rangle}\frac{\langle 2P\rangle^2}{2} \\ &= \mathrm{Split}_+(3^+1^-)A_2(2^-P^-)\,,\end{aligned} \tag{H.283}$$

with

$$\mathrm{Split}_+(3^+1^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\langle 31\rangle}\,. \tag{H.284}$$

Note that

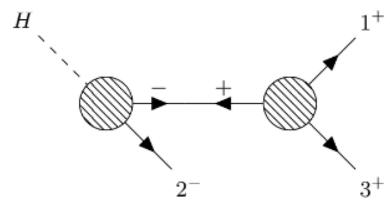


Figure H.13: $\lim_{p_1p_3\to 0} A(1^-2^+3^+)$.

$$\lim_{p_1p_3\to 0} A(1^-2^+3^+) = \lim_{p_1p_3\to 0}\Big(-\frac{\langle 13\rangle^4}{2\langle 12\rangle\langle 23\rangle\langle 31\rangle}\Big) = 0\,, \tag{H.285}$$

i. e.

$$\text{Split}_+(3^+1^+) = 0\,. \tag{H.286}$$

The four independent splitting amplitudes of the $g \to gg$ splitting are

$$\begin{cases} \text{Split}\_(3^+1^+) = \dfrac{1}{\sqrt{z(1-z)}\langle 31\rangle} \\ \text{Split}_+(3^-1^+) = \dfrac{z^2}{\sqrt{z(1-z)}\langle 31\rangle} \\ \text{Split}_+(3^+1^-) = \dfrac{(1-z)^2}{\sqrt{z(1-z)}\langle 31\rangle} \\ \text{Split}_+(3^+1^+) = 0 \end{cases}$$

The other four can be obtained by parity, e.g.

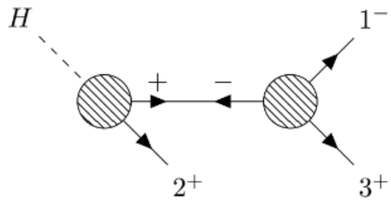


Figure H.14: $\lim\limits_{p_1p_3\to 0} A_3(1^-2^+3^+)$.

$$\begin{aligned} \lim_{p_1p_3\to 0} A_3(1^-2^+3^+) &= \frac{z^2}{\sqrt{z(1-z)}[31]}\frac{[2P]^3}{2[P2]}] \\ &= -\frac{z^2}{\sqrt{z(1-z)}[31]}\frac{[2P]^2}{2} \\ &= \text{Split}\_(3^+1^-)A_2[2^+P^+]\,, \end{aligned} \tag{H.287}$$

with

$$\text{Split}\_(3^+1^-) = -\frac{z^2}{\sqrt{z(1-z)}[31]} \tag{H.288}$$

which is the parity conjugate of $\text{Split}_+(3^-1^+)$. Likewise the others can be obtained.

Note that

$$\begin{aligned} |\text{Split}\_(3^+1^+)|^2 + |\text{Split}_+(3^-1^+)|^2 + |\text{Split}_+(3^+1^-)|^2 &\propto \frac{1+z^4+(1-z)^4}{z(1-z)} \\ &= 2\left(\frac{z}{1-z}+\frac{1-z}{z}+z(1-z)\right)\,, \end{aligned} \tag{H.289}$$

which is the z behaviour of the DGLAP $g \to gg$ splitting function.

H.26 $A_3^{\text{tree}}(1^-, 2^-, 3^+)$ from the three-gluon vertex

In sec. 1.12, we said that for $[ij] = 0$ and $s_{ij} = 0$ but $\langle ij \rangle \neq 0$ with $i, j = 1, 2, 3$,

$$iA_3^{\text{tree}}\left(1^-, 2^-, 3^+\right) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,. \tag{H.290}$$

From the colour-ordered three-gluon vertex, show that up to a sign you get indeed $A_3^{\text{tree}}(1^-, 2^-, 3^+)$ and check that it is gauge invariant.

Hint Using momentum conservation, $p_3 = -(p_1 + p_2)$, the three-gluon vertex is $V_3(p_1, p_2) = \frac{i}{\sqrt{2}}[2p_2^{\mu_1} g^{\mu_2\mu_3} - 2p_1^{\mu_2} g^{\mu_1\mu_3} + g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3}]$.

Solution. In order to check gauge invariance, it is enough to use $\epsilon_+(p_3, q_3)$ and $\epsilon_-(p_i, q_i)$, $i = 1, 2$, with arbitrary q_1, q_2, q_3. So

$$\epsilon_+(p_3, q_3) = \frac{\left\langle q_3^- \right| \gamma^\mu \left|3^-\right\rangle}{\sqrt{2}\langle q_3 3\rangle}, \qquad \epsilon_-(p_i, q_i) = -\frac{\left\langle q_i^+ \right| \gamma^\mu \left|i^+\right\rangle}{\sqrt{2}[q_i i]}\,. \tag{H.291}$$

Using the three-gluon vertex above, we get

$$\begin{aligned}
iA_3\left(1^-, 2^-, 3^+\right) &= \frac{i}{\sqrt{2}}\Big[2\left(p_2 \cdot \epsilon_-(p_1, q_1)\right)\left(\epsilon_-(p_2, q_2) \cdot \epsilon_+(p_3, q_3)\right) - 2\left(p_1 \cdot \epsilon_-(p_2, q_2)\right)\left(\epsilon_-(p_1, q_1) \cdot \epsilon_+(p_3, q_3)\right) \\
&\quad + \left(\epsilon_-(p_1, q_1) \cdot \epsilon_-(p_2, q_2)\right)\left((p_1 - p_2) \cdot \epsilon_+(p_3, q_3)\right)\Big] \\
&= \frac{i}{4}\frac{1}{[q_1 1][q_2 2]\langle q_3 3\rangle}[2\left(-[q_1 2]\langle 21\rangle\right) 2\langle q_3 2\rangle[q_2 3] - 2\left(-[q_2 1]\langle 12\rangle\right) 2\langle q_3 1\rangle[q_1 3] \\
&\quad + 2\langle 21\rangle[q_1 q_2]\left(\langle q_3 1\rangle\cancelto{0}{[13]} - \langle q_3 2\rangle\cancelto{0}{[23]}\right)] \\
&= i\frac{-[q_1 2]\langle 21\rangle\langle q_3 2\rangle[q_2 3] + [q_2 1]\langle 12\rangle\langle q_3 1\rangle[q_1 3]}{[q_1 1][q_2 2]\langle q_3 3\rangle} \\
&\overset{(1)}{=} i\frac{[q_1 3][q_2 3](\langle 31\rangle\langle q_3 2\rangle + \langle 23\rangle\langle q_3 1\rangle)}{[q_1 1][q_2 2]\langle q_3 3\rangle} \\
&\overset{(2)}{=} i\frac{[q_1 3][q_2 3]\cancel{\langle q_3 3\rangle}\langle 21\rangle}{[q_1 1][q_2 2]\cancel{\langle q_3 3\rangle}} \\
&\overset{(3)}{=} -i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,.
\end{aligned} \tag{H.292}$$

where we used momentum conservation $[q_1 2]\langle 21\rangle = -[q_1 3]\langle 31\rangle$ and $[q_2 1]\langle 12\rangle = -[q_2 3]\langle 32\rangle$ in (1), Schouten identity in (2) and again momentum conservation $[q_2 3]\langle 31\rangle = -[q_2 2]\langle 21\rangle$ and $[q_1 3]\langle 32\rangle = -[q_1 1]\langle 12\rangle$ in (3). Up to a sign (which depends on the ordering of the three gluons in the vertex), we obtain the desired result.

H.27 Colour decompositions and BCJ relations

(a) Consider the four-gluon tree amplitude. Using the BCJ relation and the multi-peripheral colour decomposition,

$$M_4^{(0)}(1,\dots,4) = g^2 \sum_{\sigma\in S_2} (F^{a_{\sigma_2}} F^{a_{\sigma_3}})_{a_1 a_4} A_4^{(0)}(1,\sigma_2,\sigma_3,4)\,, \tag{H.293}$$

write the colour decomposition of the four-gluon amplitude in terms of the colour-ordered amplitude $A_4^{(0)}(1,2,3,4)$.

Solution. Using the BCJ relation we find

$$A_4^{(0)}(1,3,2,4) = \frac{s_{12}}{s_{13}} A_4^{(0)}(1,2,3,4).$$

The multi-peripheral colour decomposition becomes

$$M_4^{(0)}(1,2,3,4) = g^2 \left[(F^{a_2}F^{a_3})_{a_1a_4} + \frac{s_{12}}{s_{13}} (F^{a_3}F^{a_2})_{a_1a_4} \right] A_4^{(0)}(1234)\,. \tag{H.294}$$

(b) Consider the five-gluon tree amplitude. Using the BCJ relations,

$$A_5^{\text{tree}}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{\text{tree}}(1,4,3,2,5)}{s_{13}s_{24}} \tag{H.295}$$

$$A_5^{\text{tree}}(1,2,4,3,5) = \frac{-s_{14}s_{25}A_5^{\text{tree}}(1,4,3,2,5) + s_{45}(s_{12}+s_{24})A_5^{\text{tree}}(1,2,3,4,5)}{s_{24}s_{35}} \tag{H.296}$$

$$A_5^{\text{tree}}(1,4,2,3,5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5) + s_{25}(s_{14}+s_{24})A_5^{\text{tree}}(1,4,3,2,5)}{s_{35}s_{24}} \tag{H.297}$$

$$A_5^{\text{tree}}(1,3,2,4,5) = \frac{-s_{14}s_{25}A_5^{\text{tree}}(1,4,3,2,5) + s_{12}(s_{24}+s_{45})A_5^{\text{tree}}(1,2,3,4,5)}{s_{13}s_{24}} \tag{H.298}$$

and the multi-peripheral colour decomposition,

$$M_5^{(0)}(1,\dots,5) = g^3 \sum_{\sigma\in S_3} (F^{a_{\sigma_2}} F^{a_{\sigma_3}} F^{a_{\sigma_4}})_{a_1 a_5} A_5^{(0)}(1,\sigma_2,\sigma_3,\sigma_4,5)\,, \tag{H.299}$$

write the colour decomposition of the five-gluon amplitudes $A_5^{(0)}(1,2,3,4,5)$, $A_5^{(0)}(1,3,2,4,5)$.

Hint Use repeatedly momentum conservation.

Solution. We consider the multi-peripheral colour decomposition of the five-gluon tree amplitude for the scattering $p_5p_1 \to p_2p_3p_4$, which displays 3! colour-ordered amplitudes. Using the BCJ relations of eqs. (H.295) - (H.298), we can write the colour-ordered amplitudes $A(12435)$, $A(13425)$, $A(14235)$ and $A(14325)$ as functions of $A(12345)$ and $A(13245)$,

$$M_5^{(0)}(1,\dots,5) = g^3 \left[c_{12345} A_5^{(0)}(12345) + c_{13245} A_5^{(0)}(13245) \right]\,, \tag{H.300}$$

where the coefficients c_{12345} and c_{13245} are each expressed as a combination of $(n-3)!\,(n-3)+1$

colour structures, in this case five colour structures,

$$
\begin{aligned}
c_{12345} &= (F^{a_2}F^{a_3}F^{a_4})_{a_1a_5} + \frac{s_{13}+s_{23}}{s_{35}}(F^{a_2}F^{a_4}F^{a_3})_{a_1a_5} + \frac{s_{12}}{s_{25}}(F^{a_3}F^{a_4}F^{a_2})_{a_1a_5} \\
&- \frac{s_{12}s_{34}}{s_{14}s_{35}}(F^{a_4}F^{a_2}F^{a_3})_{a_1a_5} + \frac{s_{12}(s_{13}-s_{25})}{s_{14}s_{25}}(F^{a_4}F^{a_3}F^{a_2})_{a_1a_5}\,, \qquad \text{(H.301)}
\end{aligned}
$$

and

$$
\begin{aligned}
c_{13245} = &= (F^{a_3}F^{a_2}F^{a_4})_{a_1a_5} + \frac{s_{12}+s_{23}}{s_{25}}(F^{a_3}F^{a_4}F^{a_2})_{a_1a_5} + \frac{s_{13}}{s_{35}}(F^{a_2}F^{a_4}F^{a_3})_{a_1a_5} \\
&- \frac{s_{13}s_{24}}{s_{14}s_{25}}(F^{a_4}F^{a_3}F^{a_2})_{a_1a_5} + \frac{s_{13}(s_{12}-s_{35})}{s_{14}s_{35}}(F^{a_4}F^{a_2}F^{a_3})_{a_1a_5}\,. \qquad \text{(H.302)}
\end{aligned}
$$

Note that c_{13245} is obtained from c_{12345} by swapping the colour and kinematic labels of gluons 2 and 3.

H.28 Proof of the Parke-Taylor formula for $j = 2$ using BCFW

For the MHV n-point sub-amplitude $A_n(1^+, \ldots, j^-, \ldots, (n-1)^+, n^-)$ with the negative-helicity gluons j and n, using on-shell recursion relation we have proven in the lecture the Parke-Taylor formula for $j > 2$.
Prove it for $j = 2$.

Hint For the shift,

$$
\begin{cases}
\hat{\lambda}_1 = \lambda_1 + z\lambda_n & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1\,, \\
\hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1 & \hat{\lambda}_n = \lambda_n
\end{cases}
$$

we have shown in the lecture that the on-shell recursion relation is reduced to the $k = 2$ case, which for $j = 2$ is

$$
iA_n(1^+, 2^-, \ldots, (n-1)^+, n^-) = iA_3(\hat{1}^+, 2^-, -\hat{P}^{-h})\frac{i}{P_{1,2}^2}iA_{n-1}(\hat{P}^h, 3^+, \ldots, (n-1)^+, \hat{n}^-)\,. \qquad \text{(H.303)}
$$

Solution. For A_{n-1} not to vanish, h must be negative,

$$
iA_n(1^+, 2^-, \ldots, (n-1)^+, n^-) = iA_3(\hat{1}^+, 2^-, -\hat{P}^+)\frac{i}{P_{1,2}^2}iA_{n-1}(\hat{P}^-, 3^+, \ldots, (n-1)^+, \hat{n}^-). \qquad \text{(H.304)}
$$

with

$$iA_3(\hat{1}^+, 2^-, -\hat{P}^+) = -i\frac{[\hat{1}\hat{P}]^4}{[\hat{1}2][2(-\hat{P})][(-\hat{P})\hat{1}]}$$
$$= i\frac{[1\hat{P}]^4}{[12][2\hat{P}][\hat{P}1]}\,. \tag{H.305}$$

and

$$iA_{n-1}(\hat{P}^-, 3^+, \ldots, (n-1)^+, \hat{n}^-) = i\frac{\langle\hat{P}n\rangle^4}{\langle\hat{P}3\rangle\ldots\langle(n-1)n\rangle\langle n\hat{P}\rangle}\,. \tag{H.306}$$

with $\hat{1}] = 1]$ and $\hat{n}\rangle = n\rangle$ since they are not shifted, and where we analytically continued $[k(-\hat{P})] = i[k\hat{P}]$. So

$$iA_n(1^+, 2^-, \ldots, (n-1)^+, n^-) = -i\frac{[1\hat{P}]^4}{[12][2\hat{P}][\hat{P}1]}\frac{1}{\langle 12\rangle[21]}\frac{\langle\hat{P}n\rangle^4}{\langle\hat{P}3\rangle\ldots\langle(n-1)n\rangle\langle n\hat{P}\rangle}$$
$$\overset{(1)}{=} -i\frac{\langle n2\rangle^4[21]^4}{\langle 34\rangle\ldots\langle(n-1)n\rangle\langle 12\rangle[21][12]\langle 23\rangle[21]\langle n1\rangle[21]}$$
$$= i\frac{\langle n2\rangle^4}{\langle 12\rangle\ldots\langle(n-1)n\rangle\langle n1\rangle}\,. \tag{H.307}$$

where in (1) we used $\langle k\hat{P}\rangle[\hat{P}m] = \langle k^-|\,\not{P} + z\not{q}\,|m^-\rangle$ together with $\langle n^-|\not{q} = \not{q}\,|1^-\rangle = 0$ on the same coloured pairs.

H.29 NMHV helicity structures

Six-gluon amplitudes display also NMHV amplitudes: the ones with three negative-helicity and three positive-helicity gluons. Up to cyclicity and reflection, there are three different helicity structures,

$$(+++---), \qquad (++-+--), \qquad (+-+-+-).$$

Using the photon decoupling identity, show that you can write the last one in terms of the first two.

Solution. For the third helicity structure, the photon decoupling identity on gluon 1 is

$$A(1^+2^-3^+4^-5^+6^-) + A(1^+3^+4^-5^+6^-2^-) + A(1^+4^-5^+6^-2^-3^+)$$
$$+A(1^+5^+6^-2^-3^+4^-) + A(1^+6^-2^-3^+4^-5^+) = 0\,. \tag{H.308}$$

We can use cyclicity on the third term and rewrite it as $A(3^+1^+4^-5^+6^-2^-)$. On the last two terms we use cyclicity and reflection and rewrite them as

$$A(5^+1^+4^-3^+2^-6^-) + A(1^+5^+4^-3^+2^-6^-)\,, \tag{H.309}$$

thus $A(1^+2^-3^+4^-5^+6^-)$ is rewritten in terms of four $(++-+--)$ helicity structures.

H.30 The $A_6(++-+--)$ NMHV amplitude

Using the on-shell recursion relation, compute the six-gluon amplitude $A_6(1^+2^+3^-4^+5^-6^-)$.

1. Using the shift,

$$\begin{cases} \hat{\lambda}_1 = \lambda_1 + z\lambda_6 & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \,, \\ \hat{\lambda}_6 = \lambda_6 & \hat{\tilde{\lambda}}_6 = \tilde{\lambda}_6 - z\tilde{\lambda}_1 \,. \end{cases} \tag{H.310}$$

Solution. The on-shell recursion relation is

$$\begin{aligned} &iA_6(1^+2^+3^-4^+5^-6^-) \\ &= \sum_{h=\pm}\sum_{k=2}^{4} iA_{k+1}\left(\hat{1}^+, 2^+, \ldots, k, -\hat{P}_{1,k}^{-h}\right)\frac{i}{P_{1,k}^2} iA_{6-k+1}\left(\hat{P}_{1,k}^h, k+1, \ldots, \hat{6}^-\right) . \end{aligned} \tag{H.311}$$

We have three cases:

$$k=2: I_2 = \begin{cases} iA_3\left(\hat{1}^+, 2^+, -\hat{P}_{1,2}^-\right)\frac{i}{P_{1,2}^2} iA_5\left(\hat{P}_{1,2}^+, 3^-, 4^+, 5^-, \hat{6}^-\right) & h=+ \\ 0 & h=- \quad A_5 \text{ vanishes}\,, \end{cases} \tag{H.312}$$

$$k=3: I_3 = \begin{cases} iA_4\left(\hat{1}^+, 2^+, 3^-, -\hat{P}_{1,3}^-\right)\frac{i}{P_{1,3}^2} iA_4\left(\hat{P}_{1,3}^+, 4^+, 5^-, \hat{6}^-\right) & h=+ \\ 0 & h=- \quad \text{both } A_4 \text{ vanish}\,, \end{cases} \tag{H.313}$$

$$k=4: I_4 = \begin{cases} iA_5\left(\hat{1}^+, 2^+, 3^-, 4^+, -\hat{P}_{1,4}^-\right)\frac{i}{P_{1,4}^2} iA_3\left(\hat{P}_{1,4}^+, 5^-, \hat{6}^-\right) & h=+ \\ 0 & h=- \quad A_5 \text{ vanishes}\,, \end{cases} \tag{H.314}$$

where I_2 and I_4 are related by parity,

$$\begin{matrix} \left|1^+\right\rangle \leftrightarrow \left|6^-\right\rangle & \left|2^+\right\rangle \leftrightarrow \left|5^-\right\rangle & \left|3^+\right\rangle \leftrightarrow \left|4^-\right\rangle \\ \left|4^+\right\rangle \leftrightarrow \left|3^-\right\rangle & \left|5^+\right\rangle \leftrightarrow \left|2^-\right\rangle & \left|6^+\right\rangle \leftrightarrow \left|1^-\right\rangle \,. \end{matrix} \tag{H.315}$$

Let us now consider the case $k=2$. For $k=2$, the pole is at

$$z_2 = -\frac{s_{12}}{\langle 6^-|\, \not{P}_{1,2}\,|1^-\rangle} = -\frac{\langle 12\rangle}{\langle 62\rangle}\,, \tag{H.316}$$

and thus

$$\hat{\not{P}}_{1,2} = \not{P}_{1,2} - \frac{\langle 12\rangle}{\langle 62\rangle}\left|6^+\right\rangle\left\langle 1^+\right|\,, \tag{H.317}$$

and $\left|\hat{1^-}\right\rangle = |1^-\rangle$ and $\left|\hat{6}^-\right\rangle = |6^-\rangle + \frac{\langle 12\rangle}{\langle 62\rangle}\left|1^-\right\rangle$. We use the identities computed in eqs. (1.417) and (1.419),

$$\langle 6\hat{P}_{1,2}\rangle[\hat{P}_{1,2}k] = \left\langle 6^-\right| \not{P}_{1,2} \left|k^-\right\rangle \,, \tag{H.318}$$

$$[5\hat{6}] = \frac{\langle 2^-|\not{p}_1 + \not{p}_6|5^-\rangle}{\langle 62\rangle}\,, \tag{H.319}$$

$$\langle 6\hat{P}_{12}\rangle[\hat{P}_{12}\hat{6}] = s_{612} = s_{345}\,. \tag{H.320}$$

Also, we use the known three particle amplitude,

$$iA_3(\hat{1}^+, 2^+, -\hat{P}_{12}) = -i\frac{[\hat{1}2]^3}{[2(-\hat{P}_{12})][(-\hat{P}_{12})\hat{1}]} = i\frac{[12]^3}{[2\hat{P}_{12}][\hat{P}_{12}1]}\,, \tag{H.321}$$

where $[(-\hat{P}_{12})k] = i[(\hat{P}_{12})k]$. Putting everything together we obtain

$$\begin{aligned}
I_2 &= i\frac{[12]^3}{[2\hat{P}_{12}][\hat{P}_{12}1]}\frac{i}{s_{12}}(-i)\frac{[\hat{P}_{12}4]^4}{[\hat{P}_{12}3][34][45][5\hat{6}][\hat{6}\hat{P}_{12}]} \\
&= i\frac{[12]^3}{[2\hat{P}_{12}][\hat{P}_{12}1]}\frac{i}{s_{12}}(-i)\frac{[\hat{P}_{12}4]^4}{[\hat{P}_{12}3][34][45][5\hat{6}][\hat{6}\hat{P}_{12}]}\cdot\frac{\langle 6\hat{P}_{12}\rangle^4}{\langle 6\hat{P}_{12}\rangle^4} \\
&= i\frac{\cancel{[12]}^3}{\cancel{\langle 62\rangle}\cancel{[21]}\langle 61\rangle\cancel{[12]}}\frac{1}{\cancel{[21]}\langle 12\rangle}\frac{\langle 6^-|\not{P}_{12}|4^-\rangle^4}{\langle 6^-|\not{P}_{12}|3^-\rangle[34][45]\frac{\langle 2^-|\not{P}_{61}|5^-\rangle}{\cancel{\langle 62\rangle}}s_{612}} \\
&= i\frac{\langle 6^-|\not{P}_{12}|4^-\rangle^4}{\langle 61\rangle\langle 12\rangle[34][45]s_{345}\langle 6^-|\not{P}_{12}|3^-\rangle\langle 2^-|\not{P}_{61}|5^-\rangle}\,,
\end{aligned} \tag{H.322}$$

and by parity we get $k = 4$,

$$I_4 = i\frac{\langle 3^-|\not{P}_{56}|1^-\rangle^4}{\langle 23\rangle\langle 34\rangle[56][61]s_{234}\langle 4^-|\not{P}_{56}|1^-\rangle\langle 2^-|\not{P}_{61}|5^-\rangle}\,. \tag{H.323}$$

For the case $k = 3$ we have

$$I_3 = i\frac{[12]^3}{[23][3\hat{P}_{13}][\hat{P}_{13}1]}\frac{i}{s_{123}}i\frac{\langle 56\rangle^3}{\langle\hat{P}_{13}4\rangle\langle 45\rangle\langle 6\hat{P}_{13}\rangle} \tag{H.324}$$

with $\hat{\not{P}}_{13} = \not{P}_{13} + z\,|6^+\rangle\langle 1^+|$, and we use

$$\langle 6\hat{P}_{13}\rangle[\hat{P}_{13}3] = \left\langle 6^-\right|\not{P}_{12}\left|3^-\right\rangle\,, \tag{H.325}$$

$$\langle 4\hat{P}_{13}\rangle[\hat{P}_{13}1] = \left\langle 4^-\right|\not{P}_{23}\left|1^-\right\rangle\,. \tag{H.326}$$

We see that we will not need the value of z. So

$$I_3 = -i\frac{[12]^3\langle 56\rangle^3}{[23]\langle 45\rangle s_{123}\langle 6^-|\not{P}_{12}|3^-\rangle\langle 4^-|\not{P}_{23}|1^-\rangle}\,. \tag{H.327}$$

Summing eqs. (H.322), (H.323) and (H.327) we obtain $A_6(1^+2^+3^-4^+5^-6^-)$.

2. Using the shift [42]

$$\begin{cases} \hat{\lambda}_4 = \lambda_4 + z\lambda_3 & \hat{\tilde{\lambda}}_4 = \tilde{\lambda}_4 \\ \hat{\lambda}_3 = \lambda_3 & \hat{\tilde{\lambda}}_3 = \tilde{\lambda}_3 - z\tilde{\lambda}_4 \end{cases} \tag{H.328}$$

Hint By cyclicity, it is convenient to take gluon 4 as the first gluon. Then the on-shell relation is

$$iA_6\left(4,5,6,1,2,3\right) = \sum_{h=\pm}\sum_{k=5}^{1} iA_{k-2}\left(4,5,\dots,k,-\hat{P}_{4,k}^{-h}\right)\frac{i}{P_{4,k}^2} iA_{6-k+4}\left(\hat{P}_{4,k}^{h},k+1,\dots,3\right), \tag{H.329}$$

where $5 \le k \le 1$ (mod 6).

Solution. k takes again three values,

$$k:5 \qquad I_5 = \sum_{h=\pm} iA_3(\hat{4}^+,5^-,-\hat{P}_{45}^{-h})\frac{i}{s_{45}} iA_5(\hat{P}_{45}^h,6^-,1^+,2^+,\hat{3}^-)\,, \tag{H.330}$$

$$k:6 \qquad I_6 = \sum_{h=\pm} iA_4(\hat{4}^+,5^-,6^-,-\hat{P}_{46}^{-h})\frac{i}{s_{456}} iA_4(\hat{P}_{46}^h,1^+,2^+,\hat{3}^-)\,, \tag{H.331}$$

$$k:1 \qquad I_1 = \sum_{h=\pm} iA_5(\hat{4}^+,5^-,6^-,1^+,-\hat{P}_{41}^{-h})\frac{i}{s_{23}} iA_3(\hat{P}_{41}^h,2^+,\hat{3}^-)\,, \tag{H.332}$$

where I_6 is only non-vanishing for $h=-$ and I_5 and I_1 are related by parity,

$$\begin{aligned} &\left|1^+\right\rangle \leftrightarrow \left|6^-\right\rangle && \left|2^+\right\rangle \leftrightarrow \left|5^-\right\rangle && \left|3^+\right\rangle \leftrightarrow \left|4^-\right\rangle\,, \\ &\left|4^+\right\rangle \leftrightarrow \left|3^-\right\rangle && \left|5^+\right\rangle \leftrightarrow \left|2^-\right\rangle && \left|6^+\right\rangle \leftrightarrow \left|1^-\right\rangle\,. \end{aligned} \tag{H.333}$$

Let us now look at the case $k=5$ with $h=+$. The three-point amplitude is only non-vanishing, if all the square brackets vanish, $[5\hat{P}] = [\hat{4}5] = [\hat{P}\hat{4}] = 0$. However, with our deformation, we have $[\hat{4}5] = [45] \neq 0$. Thus all the angle-brackets have to vanish and the three-point amplitude for $h=+$ vanishes. This leaves us with

$$I_5 = iA_3(\hat{4}^+,5^-,-\hat{P}_{45}^+)\frac{i}{s_{45}} iA_5(\hat{P}_{45}^-,6^-,1^+,2^+,\hat{3}^-) \tag{H.334}$$

where the pole is at

$$z_5 = -\frac{P_{45}^2}{2(P_{45}q)} = -\frac{P_{45}^2}{\langle 3^-|\,\not{P}_{45}\,|4^-\rangle} = -\frac{\langle 45\rangle}{\langle 35\rangle} \tag{H.335}$$

$$\Rightarrow \hat{\not{P}}_{45} = \not{P}_{45} - \frac{\langle 45\rangle}{\langle 35\rangle}\left|3^+\right\rangle\left\langle 4^+\right|. \tag{H.336}$$

We will use the identities,

$$\begin{aligned}
[2\hat{3}] &= [23] + \frac{\langle 45\rangle}{\langle 35\rangle}[24] = \frac{\langle 5^-|\, \not{P}_{34}\, |2^-\rangle}{\langle 35\rangle}\,, \\
\langle 3\hat{P}_{45}\rangle[\hat{P}_{45}k] &= \left\langle 3^-\right| \not{P}_{45} \left|k^-\right\rangle \,, \\
\langle 3\hat{P}_{45}\rangle[\hat{P}_{45}\hat{3}] &= \langle 3\hat{P}_{45}\rangle[\hat{P}_{45}3] + \frac{\langle 45\rangle}{\cancel{\langle 35\rangle}}\cancel{\langle 35\rangle}[54] = s_{345}\,,
\end{aligned} \tag{H.337}$$

which gives

$$\begin{aligned}
I_5 &= i\frac{[\hat{P}_{45}4]^3}{[45][5\hat{P}_{45}]}\frac{i}{s_{45}}(-i)\frac{[12]^3}{[\hat{P}_{45}6][61][2\hat{3}][\hat{3}\hat{P}_{45}]} \\
&= i\frac{[\hat{P}_{45}4]^3}{[45][5\hat{P}_{45}]}\frac{i}{s_{45}}(-i)\frac{[12]^3}{[\hat{P}_{45}6][61][2\hat{3}][\hat{3}\hat{P}_{45}]}\frac{\langle 3\hat{P}_{45}\rangle^3}{\langle 3\hat{P}_{45}\rangle^3} \\
&= i\frac{\langle 35\rangle^3\cancel{[54]}^3[12]^3}{\cancel{[45]}[61]\langle 45\rangle\cancel{[54]}\langle 34\rangle\cancel{[45]}\frac{\langle 5^-|\not{P}_{34}|2^-\rangle}{\langle 35\rangle}}\frac{1}{\langle 3^-|\, \not{P}_{45}\, |6^-\rangle\, s_{345}} \\
&= i\frac{\langle 35\rangle^4[12]^3}{\langle 34\rangle\langle 45\rangle[61]s_{345}\,\langle 3^-|\, \not{P}_{45}\, |6^-\rangle\,\langle 5^-|\, \not{P}_{34}\, |2^-\rangle}\,,
\end{aligned} \tag{H.338}$$

and by parity,

$$I_1 = i\frac{\langle 56\rangle^3[24]^4}{\langle 61\rangle[23][34]s_{234}\,\langle 1^-|\, \not{P}_{23}\, |4^-\rangle\,\langle 5^-|\, \not{P}_{34}\, |2^-\rangle}\,. \tag{H.339}$$

What is left, is the case $k = 6$ for which we have

$$\hat{\not{P}}_{46} = \not{P}_{46} + z_6 \left|3^+\right\rangle \left\langle 4^+\right| \,, \tag{H.340}$$

and we will use the identities,

$$\begin{aligned}
\langle 3\hat{P}\rangle[\hat{P}4] &= \left\langle 3^-\right| \not{P}_{56} \left|4^-\right\rangle \,, \\
\langle 3\hat{P}\rangle[\hat{P}6] &= \left\langle 3^-\right| \not{P}_{45} \left|6^-\right\rangle \,, \\
\langle 1\hat{P}\rangle[\hat{P}4] &= \left\langle 1^-\right| \not{P}_{56} \left|4^-\right\rangle \,,
\end{aligned} \tag{H.341}$$

to obtain

$$\begin{aligned}
I_6 &= (-i)\frac{[\hat{P}4]^3}{[45][56][6\hat{P}]}\frac{i}{s_{456}}i\frac{\langle 3\hat{P}\rangle^3}{\langle \hat{P}1\rangle\langle 12\rangle\langle 23\rangle} \\
&= (-i)\frac{[\hat{P}4]^3}{[45][56][6\hat{P}]}\frac{i}{s_{456}}i\frac{\langle 3\hat{P}\rangle^3}{\langle \hat{P}1\rangle\langle 12\rangle\langle 23\rangle}\frac{\langle 3\hat{P}\rangle[\hat{P}4]}{\langle 3\hat{P}\rangle[\hat{P}4]} \\
&= i\frac{\langle 3^-|\, \not{P}_{56}\, |4^-\rangle^4}{\langle 12\rangle\langle 23\rangle[45][56]s_{123}\,\langle 3^-|\, \not{P}_{45}\, |6^-\rangle\,\langle 1^-|\, \not{P}_{56}\, |4^-\rangle}\,.
\end{aligned} \tag{H.342}$$

To summarise our computation: For the $|6^-, 1^+\rangle$ shift we found

$$
\begin{aligned}
A_6(1^+2^+3^-4^+5^-6^-) = & \frac{\langle 6^-|\,\not{P}_{12}\,|4^-\rangle^4}{\langle 61\rangle\langle 12\rangle[34][45]s_{345}\,\langle 6^-|\,\not{P}_{12}\,|3^-\rangle\,\langle 2^-|\,\not{P}_{61}\,|5^-\rangle} \\
& + \frac{\langle 3^-|\,\not{P}_{56}\,|1^-\rangle^4}{\langle 23\rangle\langle 34\rangle[56][61]s_{234}\,\langle 4^-|\,\not{P}_{56}\,|1^-\rangle\,\langle 2^-|\,\not{P}_{61}\,|5^-\rangle} \\
& - \frac{[12]^3\langle 56\rangle^3}{[23]\langle 45\rangle s_{123}\,\langle 6^-|\,\not{P}_{12}\,|3^-\rangle\,\langle 4^-|\,\not{P}_{23}\,|1^-\rangle}\,,
\end{aligned}
\tag{H.343}
$$

whereas for the $|3^-, 4^+\rangle$ we have

$$
\begin{aligned}
A_6(1^+2^+3^-4^+5^-6^-) = & \frac{\langle 56\rangle^3[24]^4}{\langle 61\rangle[23][34]s_{234}\,\langle 1^-|\,\not{P}_{23}\,|4^-\rangle\,\langle 5^-|\,\not{P}_{34}\,|2^-\rangle} \\
& + \frac{\langle 35\rangle^4[12]^3}{\langle 34\rangle\langle 45\rangle[61]s_{345}\,\langle 3^-|\,\not{P}_{45}\,|6^-\rangle\,\langle 5^-|\,\not{P}_{34}\,|2^-\rangle} \\
& + \frac{\langle 3^-|\,\not{P}_{56}\,|4^-\rangle^4}{\langle 12\rangle\langle 23\rangle[45][56]s_{123}\,\langle 3^-|\,\not{P}_{45}\,|6^-\rangle\,\langle 1^-|\,\not{P}_{56}\,|4^-\rangle}\,.
\end{aligned}
\tag{H.344}
$$

Comparing the corresponding three-particle poles, we see that the spurious singularities are in different locations. So when in a numerical evaluation a spurious term is small in a certain region of phase space, it may be convenient in that region to evaluate the amplitude using a different shift. Keep in mind however, that **spurious** poles become an issue "only" for finite numerical precision in which the cancellation of large numbers introduces numerical instabilities. An evaluation with exact arithmetic would show that the amplitude is finite in these points and therefore the singularities are spurious. In practice, one will never use exact arithmetic but floating point evaluations and spurious singularities have to be avoided.

H.31 The $A_6(+-+-+-)$ NMHV amplitude

Although the amplitude $A_6(1^+2^-3^+4^-5^+6^-)$ can be re-written in terms of $(++-+--)$ helicity structures (see app. H.29), in order to display the singularity structure it is more convenient to work it out directly through the on-shell recursion relation. Use the shift [42],

$$
\begin{cases}
\hat{\lambda}_3 = \lambda_3 + z\lambda_2 & \hat{\tilde{\lambda}}_3 = \tilde{\lambda}_3 \\
\hat{\lambda}_2 = \lambda_2 & \hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 - z\tilde{\lambda}_3
\end{cases}
\tag{H.345}
$$

Hint By cyclicity, take 3 as the first gluon.

Solution. The on-shell recursion relation is

$$\begin{aligned} &iA_6\left(1^+,2^-,3^+,4^-,5^+,6^-\right)\\ &=\sum_{h=\pm}\sum_{k=4}^{6} iA_{k-1}\left(\hat{3}^+,4^-,\ldots,k,-\hat{P}_{3,k}^{-h}\right)\frac{i}{P_{3,k}^2}iA_{6-k+3}\left(\hat{P}_{3,k}^{h},k+1,\ldots,1^+,\hat{2}^-\right)\,. \end{aligned} \tag{H.346}$$

and k takes 3 values,

$$k:4 \qquad I_4=\sum_{h=\pm} iA_3(\hat{3}^+,4^-,-\hat{P}_{34}^{-h})\frac{i}{s_{34}}iA_5(\hat{P}_{34}^{h},5^+,6^-,1^+,\hat{2}^-)\,, \tag{H.347}$$

$$k:5 \qquad I_5=iA_4(\hat{3}^+,4^-,5^+,-\hat{P}_{35}^{-})\frac{i}{s_{345}}iA_4(\hat{P}_{35}^{+},6^-,1^+,\hat{2}^-)\,, \tag{H.348}$$

$$k:6 \qquad I_6=\sum_{h=\pm} iA_5(\hat{3}^+,4^-,5^+,6^-,-\hat{P}_{36}^{-h})\frac{i}{s_{12}}iA_3(\hat{P}_{36}^{h},1^+,\hat{2}^-)\,, \tag{H.349}$$

where I_4 and I_6 are related by parity,

$$\begin{aligned} &\left|3^+\right\rangle\leftrightarrow\left|2^-\right\rangle &&\left|1^+\right\rangle\leftrightarrow\left|4^-\right\rangle &&\left|5^+\right\rangle\leftrightarrow\left|6^-\right\rangle\\ &\left|3^-\right\rangle\leftrightarrow\left|2^+\right\rangle &&\left|1^-\right\rangle\leftrightarrow\left|4^+\right\rangle &&\left|5^-\right\rangle\leftrightarrow\left|6^+\right\rangle\,. \end{aligned} \tag{H.350}$$

Let us now consider the case $k=4$ and the helicity $h=+$. The discussion is completely analogous to the previous exercise: we see that $[\hat{3}4]=[34]\neq 0$ and therefore the amplitude has to vanish and we need consider only $h=-$. The pole is at

$$z_4=-\frac{P_{34}^2}{\langle 2^-|\,\not{p}_{34}\,|3^-\rangle}=-\frac{\langle 34\rangle}{\langle 24\rangle}\,, \tag{H.351}$$

and we have

$$\begin{aligned} &\hat{\not{P}}_{34}=\not{P}_{34}-\frac{\langle 34\rangle}{\langle 24\rangle}\left|2^+\right\rangle\left\langle 3^+\right|\\ &\left|\hat{2^-}\right\rangle=\left|2^-\right\rangle+\frac{\langle 34\rangle}{\langle 24\rangle}\left|3^-\right\rangle \qquad\qquad \left|\hat{3^-}\right\rangle=\left|3^-\right\rangle\,. \end{aligned} \tag{H.352}$$

For the computation we will use the identities,

$$\begin{aligned} [1\hat{2}]&=[12]+\frac{\langle 34\rangle}{\langle 24\rangle}[13]=\frac{\langle 4^-|\,\not{P}_{23}\,|1^-\rangle}{\langle 24\rangle}\,,\\ \langle 2\hat{P}\rangle[\hat{P}k]&=\left\langle 2^-\right|\not{P}_{34}\left|k^-\right\rangle\,,\\ \langle 2\hat{P}\rangle[\hat{P}\hat{2}]&=\langle 2\hat{P}\rangle[\hat{P}2]+\frac{\langle 34\rangle}{\langle 24\rangle}\langle 2\hat{P}\rangle[\hat{P}3]=s_{234}\,, \end{aligned} \tag{H.353}$$

and obtain

$$
\begin{aligned}
I_4 &= i\frac{[\hat{P}3]^3}{[34][4\hat{P}]}\frac{i}{s_{34}}(-i)\frac{[15]^4}{[\hat{P}5][56][61][1\hat{2}][\hat{2}\hat{P}]} \\
&= i\frac{[\hat{P}3]^3}{[34][4\hat{P}]}\frac{i}{s_{34}}(-i)\frac{[15]^4}{[\hat{P}5][56][61][1\hat{2}][\hat{2}\hat{P}]}\frac{\langle 2\hat{P}\rangle^3}{\langle 2\hat{P}\rangle^3} \\
&= i\frac{\langle 24\rangle^3\cancel{[43]^3}[15]^4}{\cancel{[34]}[56][61]\langle 34\rangle\cancel{[43]}\langle 23\rangle\cancel{[34]}\frac{\langle 4^-|\not{P}_{23}|1^-\rangle}{\langle 24\rangle}}\frac{1}{\langle 2^-|\not{P}_{34}|5^-\rangle\, s_{234}} \\
&= i\frac{\langle 24\rangle^4[15]^4}{\langle 23\rangle\langle 34\rangle[56][61]s_{234}\,\langle 2^-|\not{P}_{34}|5^-\rangle\,\langle 4^-|\not{P}_{23}|1^-\rangle}\,. \qquad (H.354)
\end{aligned}
$$

By parity we get $k=6$,

$$
I_6 = i\frac{[13]^4\langle 46\rangle^4}{[12][23]\langle 45\rangle\langle 56\rangle s_{123}\,\langle 6^-|\not{P}_{12}|3^-\rangle\,\langle 4^-|\not{P}_{23}|1^-\rangle}\,. \qquad (H.355)
$$

For the case $k=5$ we have

$$
\hat{\not{P}}_{35} = \not{P}_{35} + z_5\left|2^+\right\rangle\left\langle 3^+\right|\,, \qquad (H.356)
$$

and we need the identities,

$$
\begin{aligned}
\langle 2\hat{P}\rangle[\hat{P}k] &= \left\langle 2^-\right|\not{P}_{35}\left|k^-\right\rangle \\
\langle k\hat{P}\rangle[\hat{P}3] &= \left\langle k^-\right|\not{P}_{35}\left|3^-\right\rangle \qquad (H.357)
\end{aligned}
$$

and we will not need the value of z to obtain

$$
\begin{aligned}
I_5 &= -i\frac{[35]^4}{[34][45][5\hat{P}][\hat{P}3]}\frac{i}{s_{345}}i\frac{\langle 26\rangle^4}{\langle\hat{P}6\rangle\langle 61\rangle\langle 12\rangle\langle 2\hat{P}\rangle} \\
&= i\frac{\langle 26\rangle^4[35]^4}{\langle 61\rangle\langle 12\rangle[34][45]s_{345}\,\langle 2^-|\not{P}_{34}|5^-\rangle\,\langle 6^-|\not{P}_{45}|3^-\rangle}\,. \qquad (H.358)
\end{aligned}
$$

The amplitude is then

$$
\begin{aligned}
A_6\left(1^+,2^-,3^+,4^-,5^+,6^-\right) = &\frac{\langle 24\rangle^4[15]^4}{\langle 23\rangle\langle 34\rangle[56][61]s_{234}\,\langle 2^-|\not{P}_{34}|5^-\rangle\,\langle 4^-|\not{P}_{23}|1^-\rangle} \\
&+\frac{[13]^4\langle 46\rangle^4}{[12][23]\langle 45\rangle\langle 56\rangle s_{123}\,\langle 6^-|\not{P}_{12}|3^-\rangle\,\langle 4^-|\not{P}_{23}|1^-\rangle} \\
&+\frac{\langle 26\rangle^4[35]^4}{\langle 61\rangle\langle 12\rangle[34][45]s_{345}\,\langle 2^-|\not{P}_{34}|5^-\rangle\,\langle 6^-|\not{P}_{45}|3^-\rangle}\,. \qquad (H.359)
\end{aligned}
$$

H.32 Four spin-s boson amplitude

Consider a self-interacting massless particle of integer spin s whose three-particle amplitudes are given by (stripped off couplings)

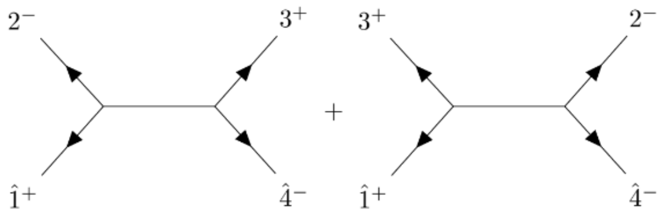
$$
\begin{aligned}
&M_3(1^-,2^-,3^+) = i\left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^s && M_3(1^+,2^+,3^-) = i(-1)^s\left(\frac{[12]^3}{[23][31]}\right)^s \\
&M_3(1^+,2^+,3^+) = 0 && M_3(1^-,2^-,3^-) = 0\,.
\end{aligned}
\tag{H.360}
$$

Using the on-shell recursion relations, compute the amplitude $M_4(1^+,2^-,3^+,4^-)$,

1. using the shift

$$
\begin{cases}
\hat{\lambda}_1 = \lambda_1 + z\lambda_4 & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \\
\hat{\tilde{\lambda}}_4 = \tilde{\lambda}_4 - z\tilde{\lambda}_1 & \hat{\lambda}_4 = \lambda_4
\end{cases}
\qquad \not{q} = \lambda_4\tilde{\lambda}_1
\tag{H.361}
$$

Hint The amplitude is unordered, so it has two contributions,



Solution. Using the shift $|4^-,1^+\rangle$ there are two contributions,

$$
\begin{aligned}
iM_4^{(4,1)} = &\underbrace{\sum_{h=\pm}\underbrace{iM_3(\hat{1}^+,2^-,-\hat{P}_{1,2}^{-h})}_{①}\frac{i}{s_{12}}iM_3(\hat{P}_{1,2}^{h},3^+,\hat{4}^-)}_{I_1} \\
&+\underbrace{\underbrace{iM_3(\hat{1}^+,3^+,-\hat{P}_{1,3}^{-})}_{②}\frac{i}{s_{13}}iM_3(\hat{P}_{1,3}^{+},2^-,\hat{4}^-)}_{I_2}\,,
\end{aligned}
\tag{H.362}
$$

where we used in the second term that $M_3(1^\pm,2^\pm,3^\pm)=0$. Since 1 and 4 must be on opposite sides of the shifted propagator, only s_{12} and s_{13} can go on-shell,

$$
\begin{aligned}
②: \qquad iM_3(\hat{1}^+,3^+,-\hat{P}_{1,3}^{-}) &= i(-1)^s\left(\frac{[13]^3}{[3(-\hat{P}_{1,3})][(-\hat{P}_{1,3})1]}\right)^s \\
&= i\left(\frac{[13]^3}{[3\hat{P}_{1,3}][\hat{P}_{1,3}1]}\right)^s,
\end{aligned}
\tag{H.363}
$$

so

$$I_2 = i\left(\frac{[13]^3}{[3\hat{P}_{1,3}][\hat{P}_{1,3}1]}\right)^s \frac{i}{s_{13}} i\left(\frac{\langle 24\rangle^3}{\langle 4\hat{P}_{1,3}\rangle\langle\hat{P}_{1,3}2\rangle}\right)^s . \tag{H.364}$$

We have

$$\hat{\not{P}}_{1,3} = \not{p}_1 + \not{p}_3 + z\left|4^+\right\rangle\left\langle 1^+\right| , \tag{H.365}$$

so

$$\begin{aligned}\langle 4\hat{P}_{1,3}\rangle[\hat{P}_{1,3}3] &= \langle 41\rangle[13] ,\\ \langle 2\hat{P}_{1,3}\rangle[\hat{P}_{1,3}1] &= \langle 23\rangle[31] ,\end{aligned} \tag{H.366}$$

and we get

$$\begin{aligned}I_2 &= -\frac{i}{s_{13}}\left(\frac{\langle 24\rangle^3[13]^{\not{3}}}{\langle 41\rangle\not{[13]}^2\langle 32\rangle}\right)^s\\ &= -i\frac{(\langle 24\rangle^2[13]^2)^s}{s_{13}}\left(\frac{\langle 24\rangle}{\langle 14\rangle\langle 23\rangle[13]}\right)^s\\ \text{(use: } \langle 23\rangle[31] = -\langle 24\rangle[41]\text{)} &= -i\frac{(\langle 24\rangle^2[13]^2)^s}{s_{13}}(s_{14})^{-s} .\end{aligned} \tag{H.367}$$

To evaluate I_1 we start with ① for the case $h = +$

$$iM_3(\hat{1}^+, 2^-, -\hat{P}_{1,2}^-) = \left(i\frac{\langle(-\hat{P}_{1,2})1\rangle}{\langle\hat{1}2\rangle\langle 2(-\hat{P}_{1,2})\rangle}\right)^s , \tag{H.368}$$

which is only non-vanishing if $[\hat{1}2] = [2(-\hat{P}_{1,2})] = [(-\hat{P}_{1,2})1] = 0$. But we have $[\hat{1}2] = [12] \neq 0$ which means $M_3(\hat{1}^+, 2^-, -\hat{P}_{1,2}^-)$ has to vanish. So we are left with

$$\begin{aligned}I_1 &= iM_3(\hat{1}^+, 2^-, -\hat{P}_{1,2}^+)\frac{i}{s_{12}} iM_3(\hat{P}_{1,2}^-, 3^+, \hat{4}^-)\\ &= i\left(\frac{[\hat{P}_{1,2}1]^3}{[12][2\hat{P}_{1,2}]}\right)^s \frac{i}{s_{12}} i\left(\frac{\langle 4\hat{P}_{1,2}\rangle}{\langle\hat{P}_{1,2}3\rangle\langle 34\rangle}\right)^s .\end{aligned} \tag{H.369}$$

with $\langle\hat{1}2\rangle = \langle 2\hat{P}\rangle = \langle\hat{P}1\rangle = [\hat{P}3] = [34] = [4\hat{P}] = 0$. Since

$$\not{P}_{1,2} = \not{p}_1 + \not{p}_2 + z\left|4^+\right\rangle\left\langle 1^+\right| , \tag{H.370}$$

we have

$$\begin{aligned}\langle k\hat{P}_{1,2}\rangle[\hat{P}_{1,2}1] &= \langle k2\rangle[21] , \qquad\qquad k = 3, 4 ,\\ \langle 4\hat{P}_{1,2}\rangle[\hat{P}2] &= \langle 41\rangle[12] .\end{aligned} \tag{H.371}$$

Multiplying and dividing I_1 by $\langle 4\hat{P}_{1,2}\rangle[\hat{P}_{12}1]$ yields

$$\begin{aligned} I_1 &= -\frac{i}{s_{12}}\left(\frac{\langle 24\rangle^4[12]^{\cancel{4}}}{\cancel{[12]}\langle 34\rangle\cancel{[12]}\langle 23\rangle\langle 41\rangle\cancel{[12]}}\right)^s \\ (\text{use: } [12]\langle 24\rangle = -[13]\langle 34\rangle) &= -\frac{i}{s_{12}}\left(\frac{[13]\langle 24\rangle^3}{\langle 23\rangle\langle 14\rangle}\right)^s \\ (\text{ same steps as for } I_2) &= -i\frac{(\langle 24\rangle^2[13]^2)^s}{s_{12}}(s_{14})^{-s}\,. \end{aligned} \tag{H.372}$$

Putting everything together, we have

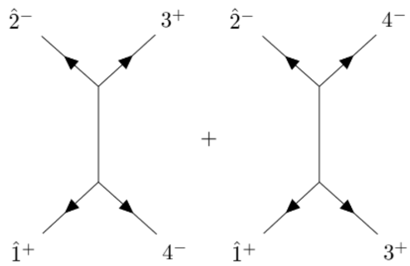
$$\begin{aligned} iM_4^{(4,1)} &= -i\left(\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)\left[\langle 24\rangle^2[13]^2\right]^s(s_{14})^{-s} \\ &= i\frac{s_{14}}{s_{12}s_{13}}\left[\langle 24\rangle^2[13]^2\right]^s(s_{14})^{-s} \\ &= i\frac{1}{s_{12}s_{13}s_{14}}\left[\langle 24\rangle^2[13]^2\right]^s(s_{14})^{-s+2}\,. \end{aligned} \tag{H.373}$$

2. using the shift

$$\begin{cases} \hat{\lambda}_1 = \lambda_1 + z\lambda_2 & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \\ \hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 - z\tilde{\lambda}_1 & \hat{\lambda}_2 = \lambda_2 \end{cases} \qquad \not{q} = \lambda_2\tilde{\lambda}_1 \tag{H.374}$$

and compare with the previous computation.

Hint In order to use it as a $|-,+\rangle$-shift, write the amplitude as $M_4(1,4,3,2)$. Also here there are two contributions,



Solution. We use the shift $|2^-, 1^+\rangle$, and by reflection we write the amplitude as $M_4(1,4,3,2)$.

$$
\begin{aligned}
iM_4^{(2,1)}(1^+,2^-,3^+,4^-) &= iM_4^{(2,1)}(1^+,4^-,3^+,2^-) \\
&= \sum_{h=\pm} \underbrace{\underbrace{iM_3(\hat{1}^+,4^-,-\hat{P}_{1,4}^{-h})}_{①} \frac{i}{s_{14}} iM_3(\hat{P}_{1,4}^{h},3^+,\hat{2}^-)}_{I_1} \\
&+ \underbrace{\underbrace{iM_3(\hat{1}^+,3^+,-\hat{P}_{1,3}^{-})}_{②} \frac{i}{s_{13}} iM_3(\hat{P}_{1,3}^{+},4^-,\hat{2}^-)}_{I_2}\,, \qquad \text{(H.375)}
\end{aligned}
$$

which can be obtained from $iM_4^{(4,1)}(1^+,2^-,3^+,4^+)$ by swapping labels 2 and 4. So

$$
iM_4^{(2,1)}(1^+,2^-,3^+,4^-) = i\frac{1}{s_{12}s_{13}s_{14}} \left[\langle 24\rangle^2[13]^2\right]^s (s_{12})^{-s+2}\,, \qquad \text{(H.376)}
$$

thus

$$
\frac{iM_4^{(4,1)}(1^+,2^-,3^+,4^-)}{iM_4^{(2,1)}(1^+,2^-,3^+,4^-)} = \left(\frac{s_{14}}{s_{12}}\right)^{2-s}\,, \qquad \text{(H.377)}
$$

and the two evaluations of $iM_4(1^+,2^-,3^+,4^-)$ may agree only if $s=2$.

H.33 Interactions of spin-1 and spin-s massless particles

We want to determine what sorts of self-consistent interactions spin $\mathbf{s}$ massless particles ($h=\pm\mathbf{s}$), that we denote by φ, can have with massless spin 1 particles, that we denote by γ [33].

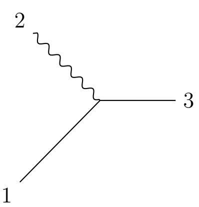


Figure H.15: 3 point interaction $\varphi\varphi\gamma$. The plain lines represent particle φ and the other type of line represents particle γ.

1. Consider the three-point amplitude $M_3(1_\varphi^{-\mathbf{s}}, 2_\gamma^{\pm}, 3_\varphi^{\mathbf{s}})$ in fig. H.15. List the corresponding helicities h_1, h_2, h_3. Derive the little group scaling of M_3 and show that

$$
\begin{aligned}
M_3(1_\varphi^{-\mathbf{s}}, 2_\gamma^{-}, 3_\varphi^{\mathbf{s}}) &\sim \langle 12\rangle^{1+2\mathbf{s}}\langle 23\rangle^{1-2\mathbf{s}}\langle 31\rangle^{-1}\,. \\
M_3(1_\varphi^{-\mathbf{s}}, 2_\gamma^{+}, 3_\varphi^{\mathbf{s}}) &\sim [12]^{1-2\mathbf{s}}[23]^{1+2\mathbf{s}}[31]^{-1} \qquad \text{(H.378)}
\end{aligned}
$$

Solution. In sec. 1.12.1, we have established that on-shell three-point amplitudes with massless particles can only depend on either right-handed or left-handed spinor products. In sec. 1.12.3, we have seen that in a generic theory little group scaling and the correct physical behaviour for real momenta imply that the three-point amplitude is made of right-handed (left-handed) spinor products if the sum of the helicities is negative (positive), eqs. (1.341) and (1.342),

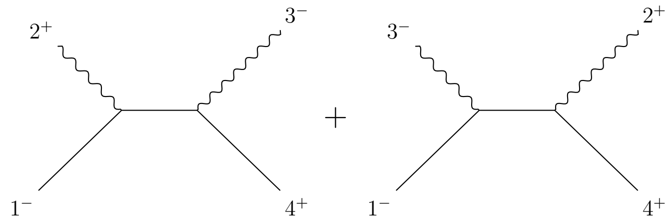
$$M_3^H \propto \langle 12\rangle^{-h_1-h_2+h_3}\langle 13\rangle^{-h_1-h_3+h_2}\langle 23\rangle^{-h_2-h_3+h_1}\Theta(-h_1-h_2-h_3)\,,$$
$$M_3^A \propto [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[13]^{h_1+h_3-h_2}\Theta(h_1+h_2+h_3)\,,$$

where the only case which is excluded is $h_1+h_2+h_3=0$.

Then setting $h_3=\mathbf{s}$, $h_2=\pm1$, $h_1=-\mathbf{s}$, we obtain straightforwardly eq. (H.378).

2. Consider the four-point amplitude $M_4(1_\varphi^-,2_\gamma^+,3_\gamma^-,4_\varphi^+)$. List all the possible diagrams built with the three-point interaction (H.378). Write down the BCFW on-shell recursion relation and compute $M_4(1_\varphi^-,2_\gamma^+,3_\gamma^-,4_\varphi^+)$.

Solution. We have the following contributions



(H.379)

that we will denote respectively by the s-channel and the u-channel in the following. By little group scaling we expect that the amplitude scales as $|1\rangle^{2\mathbf{s}}$,$|3\rangle^2$, $|4]^{2\mathbf{s}}$, $|2]^2$. First, we document the result using the channel cuts and then the on-shell recursion relations. The s-channel cut yields

$$iI_s = \frac{-i[12]^{1-2\mathbf{s}}[2P]^{1+2\mathbf{s}}}{[P1]}\frac{i}{s_{12}}\frac{i\langle(-P)3\rangle^{1+2\mathbf{s}}\langle 34\rangle^{1-2\mathbf{s}}}{\langle 4(-P)\rangle} \tag{H.380}$$

where $P=-(p_1+p_2)$. By analytic continuation $\langle(-P)q\rangle=\langle Pq\rangle$. Further,

$$[2P]\langle P3\rangle = [24]\langle 43\rangle\,,$$
$$[P1]\langle 4P\rangle = [31]\langle 43\rangle\,.$$

Then, by momentum conservation

$$[12]=\frac{[24]\langle 43\rangle}{\langle 13\rangle}\,,$$

$$I_s = (-1)^{\mathrm{s}} \frac{i}{s_{12}} ([24]\langle 13\rangle)^{2\mathrm{s}} \frac{(-1)^{2\mathrm{s}} [2|\not{p}_4|3\rangle^{2-2\mathrm{s}}}{-s_{13}}$$
$$= -(-1)^{3\mathrm{s}} \frac{i}{s_{12}} ([24]\langle 13\rangle)^{2\mathrm{s}} \frac{[2|\not{p}_4|3\rangle^{2-2\mathrm{s}}}{s_{13}} \tag{H.381}$$

The u-channel cut yields

$$iI_u = i\frac{\langle 13\rangle^{1+2\mathrm{s}}\langle 3P\rangle^{1-2\mathrm{s}}}{\langle P1\rangle} \frac{i}{s_{13}} \frac{-i[(-P)2]^{1-2\mathrm{s}}[24]^{1+2\mathrm{s}}}{[4(-P)]}, \tag{H.382}$$

where $P = -(p_1 + p_3)$. We use

$$[2P]\langle P3\rangle = [24]\langle 43\rangle,$$
$$\langle 1P\rangle[P4] = \langle 13\rangle[43].$$

and get

$$iI_u = (-1)^{\mathrm{s}} \frac{i}{s_{13}} ([24]\langle 13\rangle)^{1+2\mathrm{s}} \frac{[2|\not{p}_4|3\rangle^{1-2\mathrm{s}}}{-\langle 13\rangle[34]}$$
$$= -(-1)^{\mathrm{s}} \frac{i}{s_{13}} ([24]\langle 13\rangle)^{2\mathrm{s}} \frac{[2|\not{p}_4|3\rangle^{2-2\mathrm{s}}}{s_{12}} \tag{H.383}$$

where we multiplied and divided by $[24]\langle 43\rangle$. I_u agrees with I_s up to a factor $(-1)^{2\mathrm{s}}$.

Next, we look at the on-shell recursion relations. Consider a shift on the spin-1 particles. The shift is

$$\begin{cases} \hat{\lambda}_2 = \lambda_2 + z\lambda_3 & \hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 \\ \hat{\tilde{\lambda}}_3 = \tilde{\lambda}_3 - z\tilde{\lambda}_2 & \hat{\lambda}_3 = \lambda_3 \end{cases} \qquad \not{q} = \lambda_3\tilde{\lambda}_2 \tag{H.384}$$

The BCFW recursion relation has two contributions, the $\hat{s}-$channel and the $\hat{u}-$channel. For the first one, the two subamplitudes read

$$M_3(1^-_\varphi, \hat{2}^+_\gamma, \widehat{P}^+_\varphi) = \frac{-i[1\hat{2}]^{1-2\mathrm{s}}[\hat{2}\widehat{P}]^{1+2\mathrm{s}}}{[\widehat{P}1]},$$

evaluated at $\langle 1\hat{2}\rangle = \langle \hat{2}\hat{P}\rangle = \langle \hat{P}1\rangle = 0$, and

$$M_3(-\widehat{P}^-_\varphi, \hat{3}^-_\gamma, 4^+_\varphi) = \frac{-i\langle(-\widehat{P})\hat{3}\rangle^{1+2\mathrm{s}}\langle \hat{3}4\rangle^{1-2\mathrm{s}}}{\langle 4(-\widehat{P})\rangle},$$

evaluated at $[\hat{3}4] = [4\widehat{P}] = [\widehat{P}\hat{3}] = 0$, with $\widehat{P} = p_3 + p_4 - zq$. The pole is at

$$z = -\frac{s_{12}}{[2|\not{p}_1|3\rangle}. \tag{H.385}$$

Then

$$iI_{\hat{s}} = \frac{-i[1\hat{2}]^{1-2\mathbf{s}}[\hat{2}\widehat{P}]^{1+2\mathbf{s}}}{[\widehat{P}1]}\frac{i}{s_{12}}\frac{i\langle(-\widehat{P})\hat{3}\rangle^{1+2\mathbf{s}}\langle\hat{3}4\rangle^{1-2\mathbf{s}}}{\langle 4(-\widehat{P})\rangle}\,, \tag{H.386}$$

$$\begin{aligned}
\langle 3\hat{P}\rangle[\hat{P}2] &= \langle 34\rangle[42] \\
\langle 4\hat{P}\rangle[\hat{P}1] &= \langle 43\rangle[31] - z\langle 43\rangle[21] \\
&= [13]\langle 34\rangle - z\frac{s_{12}[12]\langle 34\rangle}{[2|\not{p}_1|3\rangle} \\
&= \frac{[13]\langle 34\rangle\langle 31\rangle[12] + s_{12}[12]\langle 34\rangle}{[2|\not{p}_1|3\rangle} \\
&= \frac{(s_{13}+s_{12})[12]\langle 34\rangle}{[12]\langle 31\rangle} \\
&= s_{14}\frac{\langle 34\rangle}{\langle 13\rangle}
\end{aligned}$$

The $\hat{s}-$channel amplitude becomes

$$iI_{\hat{s}} = -(-1)^{\mathbf{s}}\frac{i}{s_{12}}([24]\langle 13\rangle)^{2\mathbf{s}}\frac{(-1)^{2\mathbf{s}}\,[2|\not{p}_4|3\rangle^{2-2\mathbf{s}}}{s_{14}}\,. \tag{H.387}$$

The $\hat{u}$-channel amplitude yields

$$iI_{\hat{u}} = i\frac{\langle 1\hat{3}\rangle^{1+2\mathbf{s}}\langle\hat{3}\hat{P}\rangle^{1-2\mathbf{s}}}{\langle\hat{P}1\rangle}\frac{i}{s_{13}}\frac{-i[(-\hat{P})\hat{2}]^{1-2\mathbf{s}}[\hat{2}4]^{1+2\mathbf{s}}}{[4(-\hat{P})]}\,, \tag{H.388}$$

with $\hat{P} = p_2 + p_4 + zq$, and

$$z = -\frac{s_{13}}{\langle 31\rangle[12]}\,. \tag{H.389}$$

Further, similarly to before,

$$\begin{aligned}
\langle 3\hat{P}\rangle[\hat{P}2] &= \langle 34\rangle[42] \\
\langle 1\hat{P}\rangle[\hat{P}4] &= -\frac{(s_{13}+s_{12})[24]\langle 12\rangle}{[12]\langle 31\rangle} \\
&= s_{14}\frac{\langle 24\rangle}{\langle 12\rangle}
\end{aligned}$$

Hence, we find

$$\begin{aligned}
iI_{\hat{u}} &= (-1)^{\mathbf{s}}\frac{i}{s_{13}}([24]\langle 13\rangle)^{2\mathbf{s}}\frac{[2|\not{p}_4|3\rangle^{2-2\mathbf{s}}}{s_{14}}\frac{\langle 13\rangle[24][12]}{(-1)\langle 31\rangle[12][24]} \\
&= (-1)^{\mathbf{s}}\frac{i}{s_{13}}([24]\langle 13\rangle)^{2\mathbf{s}}\frac{[2|\not{p}_4|3\rangle^{2-2\mathbf{s}}}{s_{14}}
\end{aligned} \tag{H.390}$$

If $(-1)^{2\mathrm{s}} = 1$, we sum the two contributions and get

$$\begin{aligned} M_4(1^-_\varphi, 2^+_\gamma, 3^-_\gamma, 4^+_\varphi) = I_{\hat{u}} + I_{\hat{s}} &= (-1)^{\mathrm{s}} \frac{i}{s_{14}} ([24]\langle 13\rangle)^{2\mathrm{s}} [2|\not{p}_4|3\rangle^{2-2\mathrm{s}} \left(\frac{1}{s_{13}} + \frac{1}{s_{12}}\right) \\ &= (-1)^{\mathrm{s}} \frac{i}{s_{14}} ([24]\langle 13\rangle)^{2\mathrm{s}} [2|\not{p}_4|3\rangle^{2-2\mathrm{s}} \left(\frac{-s_{14}}{s_{12}s_{13}}\right) \\ &= -(-1)^{\mathrm{s}} \frac{i}{s_{12}s_{13}} ([24]\langle 13\rangle)^{2\mathrm{s}} [2|\not{p}_4|3\rangle^{2-2\mathrm{s}}, \end{aligned} \tag{H.391}$$

which is consistent with the cut-channel result.

H.34 Graviton MHV amplitudes

Tree MHV graviton amplitudes can be obtained though on-shell recursion relations, and expressed as the square of tree MHV gluon amplitudes. Let us consider the amplitude $M_n^{tree}(1^-, 2^-, 3^+, \ldots, n^+)$. Show by induction that it takes the form [30],

$$iM_n^{tree}(1^-, 2^-, 3^+, \ldots, n^+) = i \sum_{\sigma \in S_{n-2}} 2p_1 \cdot p_{\sigma_n} \Big(\prod_{k=4}^{n-1} \beta_k\Big)\Big(iA_n(1^-, 2^-, \sigma_3^+, \ldots, \sigma_n^+)\Big)^2, \tag{H.392}$$

$$\text{with} \qquad \beta_k = \begin{cases} -\dfrac{\langle \sigma_k \sigma_{k+1}\rangle}{\langle 2\sigma_{k+1}\rangle} \left\langle 2^-\right| \not{P}_{\sigma_3,\sigma_{k-1}} \left|\sigma_k^-\right\rangle & \text{for} \quad n > 4 \\ 1, & \text{for} \quad n = 4 \end{cases} \tag{H.393}$$

where $P_{i,j} = p_i + \ldots + p_j$ and $A_n(1^-, 2^-, \sigma_3^+, \ldots, \sigma_n^+)$ is the colour-ordered MHV gluon amplitude.

1. Firstly, check that the form proposed above yields the known result for $n = 4$,

$$M_4^{tree}(1^-2^-3^+4^+) = \frac{\langle 12\rangle^7 [12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}. \tag{H.394}$$

Solution. For $n = 4$, the proposed form yields

$$\begin{aligned} iM_4^{tree}(1^-, 2^-, 3^+, 4^+) &= i\big[s_{14}(iA_4(1^-, 2^-, 3^+, 4^+))^2 + s_{13}(iA_4(1^-, 2^-, 4^+, 3^+))^2\big] \\ &= -i\Big[\frac{[14]\cancel{\langle 41\rangle}\langle 12\rangle^6}{\langle 23\rangle^2\langle 34\rangle^2\langle 41\rangle^{\cancel{2}}} + \frac{[13]\cancel{\langle 31\rangle}\langle 12\rangle^6}{\langle 24\rangle^2\langle 43\rangle^2\langle 31\rangle^{\cancel{2}}}\Big] \\ &= -i\frac{\langle 12\rangle^6([14]\langle 24\rangle^2\langle 31\rangle + [13]\langle 23\rangle^2\langle 41\rangle)}{\langle 13\rangle\langle 14\rangle\langle 23\rangle^2\langle 24\rangle^2\langle 34\rangle^2} \\ &= -i\frac{\langle 12\rangle^6\cancel{\langle 23\rangle}[13](-\langle 24\rangle\langle 31\rangle + \langle 23\rangle^2\langle 41\rangle)}{\langle 13\rangle\langle 14\rangle\langle 23\rangle^{\cancel{2}}\langle 24\rangle^2\langle 34\rangle^2} \\ &= -i\frac{\langle 12\rangle^7[13]\langle 34\rangle}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle^2\langle 34\rangle^2} \\ &= i\frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}, \end{aligned} \tag{H.395}$$

where in the fourth line we used momentum conservation $\langle 24\rangle[41] = -\langle 23\rangle[31]$, in the fifth line the Schouten identity, and then finally $[13]\langle 34\rangle = -[12]\langle 24\rangle$.

2. Next, assume that eq. (H.392) holds for $(n-1)$ gravitons and consider the shift,

$$\begin{cases} \hat{\lambda}_1 = \lambda_1 - z\lambda_2 & \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \\ \hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 + z\tilde{\lambda}_1 & \hat{\lambda}_2 = \lambda_2\,. \end{cases} \tag{H.396}$$

Eq. (H.396) is a $|-,-\rangle$ shift. It has been shown that under the shifts $|-,-\rangle$, $|+,+\rangle$, $|-,+\rangle$, $M(z) \to 1/z^2$ as $z \to \infty$, so $|-,-\rangle$ is a good shift [32].

Use the on-shell recursion relation for graviton amplitudes,

$$\begin{aligned} &iM_n(1^-,2^-,3^+,\ldots,n^+) \\ &= \sum_{\sigma\in\mathbb{Z}_{n-2}}\sum_{h=\pm}\sum_{k=3}^{n-1} iM_k(\hat{2}^-,\sigma_3^+,\ldots,\sigma_n^+,\hat{P}_{2,k}^h)\frac{i}{P_{2,k}^2}\, iM_{n-k+2}(-\hat{P}_{2,k}^{-h},\sigma_{k+1}^+,\ldots,\sigma_n^+,\hat{1}^-)\,, \end{aligned} \tag{H.397}$$

where the first sum is over the $(n-2)$ cyclic permutations on the helicity line $(3^+,\ldots,n^+)$, considering that the graviton amplitudes are not coloured ordered (there is no colour in gravity).

Hint. Since we want to write the graviton amplitudes as the square of gluon amplitudes, we establish the analogous on-shell recursion relation for gluon amplitudes under the same shift,

$$\begin{aligned} &iA_n(1^-,2^-,3^+,\ldots,n^+) \\ &= \sum_{h=\pm}\sum_{k=3}^{n-1} iA_k(\hat{2}^-,\sigma_3^+,\ldots,\sigma_n^+,\hat{P}_{2,k}^h)\frac{i}{P_{2,k}^2}\, iA_{n-k+2}(-\hat{P}_{2,k}^{-h},(k+1)^+,\ldots,n^+,\hat{1}^-)\,. \end{aligned} \tag{H.398}$$

Solution. For the gluon amplitudes we use the on-shell recursion relations (H.398). As we have discussed in the proof of the Parke Taylor formula with BCFW, we can discard all the terms with $4 \le k \le n-2$ by counting the negative helicities. We are left with the $k = 3$ and $k = n-1$ terms. For $k = 3$, we have

$$I_3 = iA_3(\hat{2}^-,3^+,\hat{P}_{2,3}^+)\frac{i}{s_{23}} iA_{n-1}(-\hat{P}_{2,3}^-,4^+,n^+,\hat{1}^-)\,. \tag{H.399}$$

The 3-point amplitude,

$$iA_3(\hat{2}^-,3^+,\hat{P}_{2,3}^+) = -i\frac{[3\hat{P}]^3}{[\hat{2}3][\hat{P}\hat{2}]}\,, \tag{H.400}$$

can be non-vanishing only if

$$\langle 3\hat{P}\rangle = \langle\hat{2}3\rangle = \langle\hat{P}2\rangle = 0\,, \tag{H.401}$$

but $\langle\hat{2}3\rangle = \langle 23\rangle \ne 0$, and this leads to $[3\hat{P}] = [\hat{2}3] = [\hat{P}2] = 0$ and thus $A_3(\hat{2}^-,3^+,\hat{P}_{2,3}^+) = 0$. Thus, we are left with only one term, $k = n-1$,

$$iA_n(1^-,2^-,3^+,\ldots,n^+) = iA_{n-1}(\hat{2}^-,3^+,\ldots,(n-1)^+,\hat{P}_{1,n}^-)\frac{i}{s_{n1}} iA_3(-\hat{P}_{1,n}^+,n^+,\hat{1}^-)\,. \tag{H.402}$$

Since the graviton amplitudes with all-like helicity gravitons, or all but one, vanish, the analogous analysis of the on-shell recursion relation of the MHV graviton amplitudes yields just one term, for $k = n-1$, up to cyclic permutations, since the graviton amplitudes are not colour-ordered,

$$iM_n(1^-, 2^-, 3^+, \ldots, n^+) = \sum_{\sigma\in\mathbb{Z}_{n-2}} iM_{n-1}(\hat{2}^-, \sigma_3^+, \ldots, \sigma_{(n-1)}^+, \hat{P}_{1,n}^-)\frac{i}{s_{\sigma_n 1}} iM_3(-\hat{P}_{1,n}^+, \sigma_n^+, \hat{1}^-)\,. \tag{H.403}$$

We may symmetrise over the $(n-3)!$ non-cyclic permutations, and write it as

$$\begin{aligned} & iM_n(1^-, 2^-, 3^+, \ldots, n^+) \\ &= \frac{1}{(n-3)!}\sum_{\sigma\in\mathbb{S}_{n-2}} iM_{n-1}(\hat{2}^-, \sigma_3^+, \ldots, \sigma_{(n-1)}^+, \hat{P}_{1,\sigma_n}^-)\frac{i}{s_{\sigma_n 1}} iM_3(-\hat{P}_{1,\sigma_n}^+, \sigma_n^+, \hat{1}^-)\,. \end{aligned} \tag{H.404}$$

Then, we use the ansatz for M_{n-1},

$$\begin{aligned} & iM_n(1^-, 2^-, 3^+, \ldots, n^+) \\ &= \frac{1}{(n-3)!}\sum_{\sigma\in S_{n-2}} i \sum_{\sigma\in S_{n-3}} 2\hat{P}_{1,\sigma_n}\cdot p_{\sigma_{n-1}}\Big(\prod_{k=4}^{n-2}\beta_k\Big)\Big(iA_n(\hat{2}^-, \sigma_3^+, \ldots, \sigma_{n-1}^+, \hat{P}_{1,\sigma_n}^-)\Big)^2 \\ &\quad \cdot\frac{i}{s_{1\sigma_n}} i\Big(iA_3(-\hat{P}_{1,\sigma_n}^+, \sigma_n^+, \hat{1}^-)\Big)^2\,, \end{aligned} \tag{H.405}$$

the sum over S_{n-3} is redundant and may be cancelled with the symmetry factor $(n-3)!$. Then we use the on-shell recursion for gluons (H.402),

$$iA_{n-1}(\hat{2}^-, 3^+, \ldots, n-1^+, \hat{P}_{1,n}^-)\; iA_3(-\hat{P}_{1,n}^+, n^+, \hat{1}^-) = -is_{n1}\cdot iA_n(1^-, 2^-, 3^+, \ldots, n^+)\,, \tag{H.406}$$

so that

$$iM_n(1^-, 2^-, 3^+, \ldots, n^+) = i\sum_{\sigma\in S_{n-2}} 2\hat{P}_{1,\sigma_n}\cdot p_{\sigma_{n-1}} 2p_1\cdot p_{\sigma_n}\Big(\prod_{k=4}^{n-2}\beta_k\Big)\Big(iA_n(1^-, 2^-, 3^+, \ldots, n^+)\Big)^2\,. \tag{H.407}$$

We need to work out $2\hat{P}_{1,\sigma_n}\cdot p_{\sigma_{n-1}}$. Firstly, we need the value of z at the pole,

$$\begin{aligned} 0 = \hat{P}_{1,\sigma_n}^2 &= [\hat{1}\sigma_n]\langle\sigma_n\hat{1}\rangle \\ &= [1\sigma_n]\big(\langle\sigma_n 1\rangle - z\langle\sigma_n 2\rangle\big)\,, \end{aligned} \tag{H.408}$$

which entails that

$$z = \frac{\langle\sigma_n 1\rangle}{\langle\sigma_n 2\rangle}\,. \tag{H.409}$$

$$
\begin{aligned}
2\hat{P}_{1,\sigma_n}\cdot p_{\sigma_{n-1}} &= 2p_{\sigma_{n-1}}\cdot(\hat{p}_1+p_{\sigma_n}) \\
&= [1\sigma_{n-1}]\langle\sigma_{n-1}\hat{1}\rangle + [\sigma_n\sigma_{n-1}]\langle\sigma_{n-1}\sigma_n\rangle \\
&= [1\sigma_{n-1}]\big(\langle\sigma_{n-1}1\rangle - \frac{\langle\sigma_n 1\rangle}{\langle\sigma_n 2\rangle}\langle\sigma_{n-1}2\rangle\big) + [\sigma_n\sigma_{n-1}]\langle\sigma_{n-1}\sigma_n\rangle \\
&= [1\sigma_{n-1}]\frac{\langle\sigma_{n-1}\sigma_n\rangle\langle 12\rangle}{\langle\sigma_n 2\rangle} + [\sigma_n\sigma_{n-1}]\langle\sigma_{n-1}\sigma_n\rangle \\
&= \frac{\langle\sigma_{n-1}\sigma_n\rangle}{\langle 2\sigma_n\rangle}\big(\left\langle 2^-\right|\not{1}\left|\sigma_{n-1}^-\right\rangle + \left\langle 2^-\right|\not{\sigma}_n\left|\sigma_{n-1}^-\right\rangle\big) \\
&= -\frac{\langle\sigma_{n-1}\sigma_n\rangle}{\langle 2\sigma_n\rangle}\left\langle 2^-\right|\not{P}_{2,\sigma_{n-1}}\left|\sigma_{n-1}^-\right\rangle \\
&= -\frac{\langle\sigma_{n-1}\sigma_n\rangle}{\langle 2\sigma_n\rangle}\left\langle 2^-\right|\not{P}_{\sigma_3,\sigma_{n-2}}\left|\sigma_{n-1}^-\right\rangle \\
&= \beta_{n-1}\,,
\end{aligned} \tag{H.410}
$$

where in the fourth line we used the Schouten identity, in the sixth line momentum conservation, and in the last line eq. (H.393). Substituting eq. (H.410) into eq. (H.407), we obtain eq. (1.440) and this completes the proof.

H.35 Five-graviton MHV amplitude

Write the n-graviton MHV the amplitude M_n as

$$
M_n(1,2,\ldots,n) = \langle ij\rangle^8\tilde{M}_n(1,2,\ldots,n)\,, \tag{H.411}
$$

where i and j are the negative-helicity gravitons and $\tilde{M}_n$ is helicity independent. Use Hodges formula [31] for $\tilde{M}_n$

$$
\tilde{M}_n(1,2,\ldots,n) = (-1)^{n+1}\mathrm{sgn}(ijk)\ \mathrm{sgn}(rst)c_{ijk}c^{rst}|\phi|^{ijk}_{rst}\,, \tag{H.412}
$$

where $\mathrm{sgn}(ijk) \equiv \mathrm{sgn}(\sigma(i,j,k,1,2,\ldots,\not{i},\not{j},\not{k},\ldots,n))$ is the signature of the permutation which moves i,j,k up front in the sequence. Compute $\tilde{M}_5$ using the minor determinants,

1. $|\phi|^{123}_{123}$

   Solution. We recall that

$$
\begin{cases}
\phi^i{}_j = \dfrac{[ij]}{\langle ij\rangle} \qquad j\neq i\,, \\
\phi^i{}_i = -\displaystyle\sum_{k\neq i}\frac{[ik]\langle kx\rangle\langle ky\rangle}{\langle ik\rangle\langle ix\rangle\langle iy\rangle}\,.
\end{cases} \tag{H.413}
$$

   and the minor determinant $|\phi|^{ijk}_{pqr}$ is obtained by deleting rows i,j,k and columns p,q,r from the matrix whose entries are $\phi^i{}_j$ above. Further, the coefficients read

$$
c^{ijk} = c_{ijk} = (\langle ij\rangle\langle jk\rangle\langle ki\rangle)^{-1}\,. \tag{H.414}
$$

Therefore, we have

$$\tilde{M}_5 = \frac{\varphi^4_{[4}\varphi^5_{5]}}{(\langle 12\rangle\langle 23\rangle\langle 31\rangle)^2} \tag{H.415}$$

with

$$\varphi^4_4 = -\frac{[45]\langle 51\rangle\langle 52\rangle}{\langle 45\rangle\langle 51\rangle\langle 52\rangle} - \frac{[43]\langle 31\rangle\langle 32\rangle}{\langle 43\rangle\langle 41\rangle\langle 42\rangle} \tag{H.416}$$

$$\varphi^5_5 = -\frac{[54]\langle 41\rangle\langle 42\rangle}{\langle 54\rangle\langle 51\rangle\langle 52\rangle} - \frac{[53]\langle 31\rangle\langle 32\rangle}{\langle 54\rangle\langle 51\rangle\langle 52\rangle} \tag{H.417}$$

$$\varphi^4_5 = \varphi^5_4 = \frac{[45]}{\langle 45\rangle}\,. \tag{H.418}$$

We use the explicit expression in

$$\begin{aligned}\varphi^4_{[4}\varphi^5_{5]} = &\left(\frac{[45]}{\langle 45\rangle}\right)^2\left(\frac{\langle 51\rangle\langle 52\rangle\langle 41\rangle\langle 42\rangle}{\langle 41\rangle\langle 42\rangle\langle 51\rangle\langle 52\rangle} - 1\right)\\ &+\frac{[54]\langle 41\rangle\langle 42\rangle}{\langle 54\rangle\langle 51\rangle\langle 52\rangle}\frac{[43]\langle 31\rangle\langle 32\rangle}{\langle 43\rangle\langle 41\rangle\langle 42\rangle}\\ &+\frac{[53]\langle 31\rangle\langle 32\rangle}{\langle 54\rangle\langle 51\rangle\langle 52\rangle}\frac{[45]\langle 51\rangle\langle 52\rangle}{\langle 45\rangle\langle 51\rangle\langle 52\rangle}\\ &+\frac{[53]\langle 31\rangle\langle 32\rangle}{\langle 54\rangle\langle 51\rangle\langle 52\rangle}\frac{[43]\langle 31\rangle\langle 32\rangle}{\langle 43\rangle\langle 41\rangle\langle 42\rangle}\end{aligned} \tag{H.419}$$

Performing the simplifications with c_{123} we are left with

$$\begin{aligned}\tilde{M}_5 = \frac{1}{\langle 12\rangle^2}\Bigg(&\frac{[43][53]}{\langle 43\rangle\langle 41\rangle\langle 42\rangle\langle 53\rangle\langle 51\rangle\langle 52\rangle}\\ &+\frac{[45][53]}{\langle 23\rangle\langle 31\rangle\langle 45\rangle\langle 41\rangle\langle 42\rangle\langle 53\rangle}\\ &+\frac{[54][43]}{\langle 54\rangle\langle 51\rangle\langle 52\rangle\langle 43\rangle\langle 31\rangle\langle 23\rangle}\Bigg)\end{aligned} \tag{H.420}$$

2. $|\phi|^{345}_{123}$ and verify that they agree.

Solution. We start with

$$\tilde{M}_5 = -\frac{\varphi^1_4\varphi^2_5 - \varphi^1_5\varphi^2_4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle\langle 34\rangle\langle 45\rangle\langle 53\rangle}\,, \tag{H.421}$$

where

$$\varphi^1_4 = \frac{[14]}{\langle 14\rangle}\,, \varphi^1_5 = \frac{[15]}{\langle 15\rangle}\,, \varphi^2_4 = \frac{[24]}{\langle 24\rangle}\,, \varphi^2_5 = \frac{[25]}{\langle 25\rangle}\,, \tag{H.422}$$

$$\varphi^1_4\varphi^2_5 - \varphi^1_5\varphi^2_4 = \frac{[14][25]\langle 24\rangle\langle 15\rangle - [15][24]\langle 14\rangle\langle 25\rangle}{\langle 14\rangle\langle 25\rangle\langle 24\rangle\langle 15\rangle}. \tag{H.423}$$

Using momentum conservation one can verify that the two expressions correspond.

H.36 Gravity with light and matter

In order to investigate gravitational interactions with light and matter, consider an extension of the Einstein-Hilbert action, where we add the minimal coupling of gravity to a massive scalar field [6].

$$\mathcal{S}_{HE} \equiv \int d^4x\sqrt{-g}\left\{\frac{2}{\kappa^2}R - \frac{g^{\mu\nu}g^{\rho\sigma}}{4}F_{\mu\rho}F_{\nu\sigma} + \frac{g^{\mu\nu}}{2}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - \frac{m^2}{2}\phi^2\right\}, \tag{H.424}$$

where $g^{\mu\nu}$ is the metric tensor, g its determinant, R is the Ricci curvature and κ is a coupling constant related to Newton's constant by $\kappa^2 = 32\pi G_N$.

As already seen in the lectures, the rules for the propagators and vertices involving gravitons are obtained by expanding the metric around flat space (1.425), $g^{\mu\nu} = \eta^{\mu\nu} + \kappa \mathrm{h}^{\mu\nu}$. However, we bypass the extraction of the Feynman rules from the Lagrangian and use the spinor-helicity formalism, combined to the BCFW recursion, in order to calculate some amplitudes that have relevance in gravitational physics: the gravitational bending of light by a mass, and the scattering of a gravitational wave off a mass. In both examples, the mass acting as a source of gravitational field is taken to be a scalar particle. External scalar particles must have helicity $h = 0$[1], photons can have helicities $h = \pm 1$ and gravitons can have helicities $h = \pm 2$. One may choose a gauge in which their polarisation vectors are "squares" of the gluon polarisation vectors (1.427), $\epsilon^{\mu\nu}_{2h}(p,q) = \epsilon^{\mu}_{h}(p,q)\epsilon^{\nu}_{h}(p,q)$.

1. For three-point amplitudes that involve only massless particles (photons and gravitons), little group scaling and dimensional analysis are sufficient to constrain completely their form. What is the mass dimension expected for M_3? What is the coupling for the three-point vertex h$\gamma\gamma$? Derive the expression of

$$M_3\left(1^{\pm}_{\mathrm{h}}, 2^{+}_{\gamma}, 3^{+}_{\gamma}\right) \quad M_3\left(1^{+}_{\mathrm{h}}, 2^{+}_{\gamma}, 3^{-}_{\gamma}\right) \quad M_3\left(1^{+}_{\mathrm{h}}, 2^{-}_{\gamma}, 3^{-}_{\gamma}\right)$$
$$M_3\left(1^{-}_{\mathrm{h}}, 2^{+}_{\gamma}, 3^{-}_{\gamma}\right) \quad M_3\left(1^{-}_{\mathrm{h}}, 2^{-}_{\gamma}, 3^{+}_{\gamma}\right)$$

Solution. We use the results of eqs. (1.341) and (1.342) with $h_1 = \pm 2$, $h_2 = \pm 1$, $h_3 = \pm 1$, considering in addition that M_3 must have overall mass dimension 1 and that the coupling κ has mass dimension -1. We obtain

$$M_3\left(1^{\pm}_{\mathrm{h}}, 2^{+}_{\gamma}, 3^{+}_{\gamma}\right) = M_3\left(1^{\pm}_{\mathrm{h}}, 2^{-}_{\gamma}, 3^{-}_{\gamma}\right) = 0 \tag{H.425}$$
$$M_3\left(1^{+}_{\mathrm{h}}, 2^{+}_{\gamma}, 3^{-}_{\gamma}\right) = \frac{\kappa}{2}[12]^4[23]^{-2} \tag{H.426}$$
$$M_3\left(1^{+}_{\mathrm{h}}, 2^{-}_{\gamma}, 3^{-}_{\gamma}\right) = \frac{\kappa}{2}[23]^{-2}[31]^4 \tag{H.427}$$
$$M_3\left(1^{-}_{\mathrm{h}}, 2^{+}_{\gamma}, 3^{-}_{\gamma}\right) = \frac{\kappa}{2}\langle 23\rangle^{-2}\langle 31\rangle^4 \tag{H.428}$$
$$M_3\left(1^{-}_{\mathrm{h}}, 2^{-}_{\gamma}, 3^{+}_{\gamma}\right) = \frac{\kappa}{2}\langle 12\rangle^4\langle 23\rangle^{-2} \tag{H.429}$$

[1]For a massive scalar particle, we would need to consider the representation of the little group of spin 0, i.e. the singlet. However, for conciseness the amplitudes with massive scalar particles do not carry explicit little group indices.

2. For the $\phi\phi$h amplitude, we cannot rely on little group scaling because the massive scalar field is in the singlet representation of SU(2), thus it does not change under the action of SU(2). Instead, we turn to the Feynman rule for the corresponding vertex $\Gamma^{\mu\nu}$, which reads [6]

$$\Gamma^{\mu\nu}(p_1, p_2) = \frac{i\kappa}{2}\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \eta^{\mu\nu}\left(p_1 \cdot p_2 - m^2\right)\right], \tag{H.430}$$

Where p_1, p_2 are the momenta of the legs of particle ϕ. This vertex is intendend to be contracted with the polarization vector of the graviton to give $\epsilon^*_{\mu\nu}\Gamma^{\mu\nu} = iM_{\phi\phi\text{h}}$. Derive

$$M_3(1_\phi, 2_\phi, 3_\text{h}^+), \quad M_3(1_\phi, 2_\phi, 3_\text{h}^-). \tag{H.431}$$

Solution. In the following we will denote ϵ^* as ϵ for brevity. The term proportional to $\eta^{\mu\nu}$ gives the contraction $\epsilon^\pm(p_3, q) \cdot \epsilon^\pm(p_3, q)$ which vanishes by Fierz rearrengement. Hence,

$$\begin{aligned} M_3(1_\phi, 2_\phi, 3_\text{h}^+) &= i\left(p_1 \cdot \epsilon_+(p_3; q)\right)\left(p_2 \cdot \epsilon_+(p_3; q)\right) \\ &= \frac{i\kappa}{2}\frac{\left\langle q\left|\not{p}_1\right|p_3\right]\left\langle q\left|\not{p}_2\right|p_3\right]}{\langle qp_3\rangle^2} \end{aligned} \tag{H.432}$$

$$M_3(1_\phi, 2_\phi, 3_\text{h}^-) = \frac{i\kappa}{2}\frac{\left\langle p_3\left|\not{p}_1\right|q\right]\left\langle p_3\left|\not{p}_2\right|q\right]}{[qp_3]^2}. \tag{H.433}$$

H.37 Gravitational bending of light

Using on-shell recursion relations, derive the amplitude $M_4(1_\gamma^+, 2_\gamma^-, 3_\phi, 4_\phi)$ for the scattering between a photon and a massive scalar, mediated by a graviton. Since the photon γ and the scalar ϕ do not couple directly, the only tree diagram involves the exchange of an intermediate graviton. As a consequence, the only possible shift for a BCFW recursion relation involves γ and ϕ, since the shifted legs need to be on opposite sides of the internal propagator. Consider the BCFW relation with the shift,

$$\widehat{p}_2 \equiv p_2 + zq, \qquad \widehat{p}_3 \equiv p_3 - zq. \tag{H.434}$$

1. Show that the vector q^μ such that

$$|q\rangle = |2\rangle \tag{H.435}$$

$$|q] = \not{p}_3\,|2\rangle \tag{H.436}$$

satisfies the requirements to be a good shift, i.e.

$$q^2 = 0, \quad q \cdot p_2 = q \cdot p_3 = 0. \tag{H.437}$$

Solution. Since p_2 is lightlike, the condition $2p_2 \cdot q = \langle 2q\rangle[q2] = \langle 2^-|\,|q^+\rangle\,\langle q^+|2^-\rangle = 0$ can be satisfied by choosing, for instance,

$$|q\rangle = |2\rangle.$$

We turn to the other condition,

$$2q \cdot p_3 = \langle q|\not{p}_3|q] = \langle 2|\not{p}_3|q] \tag{H.438}$$

which can be satisfied by $|q] = \not{p}_3|2\rangle$,

$$\langle 2|\not{p}_3|q] = \langle 2|\not{p}_3\not{p}_3|2\rangle = m^2\langle 22\rangle = 0. \tag{H.439}$$

2. Assume $M_4(z) \to 0$ as $z \to \infty$ and compute $M_4(1^+_\gamma, 2^-_\gamma, 3_\phi, 4_\phi)$ starting from the recursion relation,

$$iM_4(1^+_\gamma, 2^-_\gamma, 3_\phi, 4_\phi) = \sum_h iM_3(1^+_\gamma, \hat{2}^-_\gamma, -\hat{P}^{+h}_{\text{h}})\frac{i}{P^2}iM_3(\hat{P}^{-h}_{\text{h}}, \hat{3}_\phi, 4_\phi) \tag{H.440}$$

Hint. *A convenient choice of reference for the polarisation vector is $q = p_2$.*

Hint. *Find explicitly the value of z such that $\hat{P}^2(z) = 0$ in order to appreciate the differences from a massless shift.*

Solution. As opposed to the massless shift, here we don't have an explicit expression of the shift q^μ via the Gordon identity, because the massive momentum shift does not factorize into singular shifts of the kets associated to it. Nonetheless, from eq. (H.435) and (H.436), it is natural to consider the operator $|q]\,\langle q|$. If we contract eq. (H.434) with γ_μ and then consider only the operator $(p\cdot\sigma)^{\dot{b}b} = (|p]\,\langle p|)^{\dot{b}b}$, we have

$$\left|\hat{2}\right]\left\langle\hat{2}\right| = |2]\,\langle 2| + z\,|q]\,\langle q| = |2]\,\langle 2| + z\not{p}_3\,|2\rangle\,\langle 2| = (|2] + z\not{p}_3\,|2\rangle)\,\langle 2|\,, \tag{H.441}$$

$$\hat{p}_3\cdot\sigma = p_3\cdot\sigma - z\,|q]\,\langle q| = p_3\cdot\sigma - z\not{p}_3\,|2\rangle\,\langle 2|\;. \tag{H.442}$$

We see from eq. (H.441) that

$$\left|\hat{2}\right] = |2] + z\not{p}_3\,|2\rangle \tag{H.443}$$

$$\left|\hat{2}\right\rangle = |2\rangle \tag{H.444}$$

and, as already anticipated, the same factorization cannot be performed in $\hat{p}_3\cdot\sigma$. We need to derive the value of z such that $0 = \widehat{P}^2_{1,2}(z)$. We have

$$0 = \widehat{P}^2_{1,2}(z) = 2\hat{p}_2(z)\cdot p_1 = [1\hat{2}]\langle\hat{2}1\rangle = \langle 21\rangle\,[1|\left(|2] + z\not{p}_3\,|2\rangle\right), \tag{H.445}$$

which implies

$$z = -\frac{[12]}{[1|\not{p}_3\,|2\rangle} \tag{H.446}$$

Now we write the recursion relation. Plugging in the results for the three-point amplitudes and summing explicitly over the two helicities of the intermediate graviton, the $\gamma\gamma\phi\phi$ amplitude can be written as

$$iM_4(1_\gamma^+, 2_\gamma^-, 3_\phi, 4_\phi) = \frac{\kappa^2}{4}\frac{1}{\langle 12\rangle[12]}\left\{\frac{\left[\widehat{P}_{1,2}|1\right]^4}{[1\hat{2}]^2}\frac{\left\langle \widehat{P}_{1,2}\left|\not{p}_4\right| q\right]^2}{\left[q|\widehat{P}_{1,2}\right]^2} + \frac{\left\langle \widehat{P}_{1,2}|\hat{2}\right\rangle^4}{\langle 1\hat{2}\rangle^2}\frac{\left\langle q\left|\not{p}_4\right|\widehat{P}_{1,2}\right]^2}{\left\langle q|\widehat{P}_{1,2}\right\rangle^2}\right\}. \quad \text{(H.447)}$$

In M_4 above q^μ is the reference vector of polarisation vector of the graviton[2]. For the first term, we may write

$$\frac{\left[\widehat{P}_{1,2}|1\right]^4}{[1\hat{2}]^2} = \frac{\left[\widehat{P}_{1,2}|1\right]^4\left\langle \widehat{P}_{1,2}|1\right\rangle^4}{[1\hat{2}]^2\left\langle \widehat{P}_{1,2}|1\right\rangle^4} = \frac{(2p_1\cdot\hat{p}_2)^4}{[1\hat{2}]^2\left\langle \widehat{P}_{1,2}|1\right\rangle^4} = 0,$$

since we evaluate the expression at z. Therefore, the amplitude reduces to a single term and we set the reference of the polarization vector to $q = p_2$, which leads to

$$iM_4(1_\gamma^+, 2_\gamma^-, 3_\phi, 4_\phi) = \frac{\kappa^2}{4}\frac{\left\langle \widehat{P}_{1,2}|2\right\rangle^2\left\langle 2\left|\not{p}_4\right|\widehat{P}_{1,2}\right]^2}{\langle 12\rangle^3[12]}. \quad \text{(H.448)}$$

Then, notice that

$$\left\langle 2\left|\not{p}_4\right|\widehat{P}_{1,2}\right]\left\langle \widehat{P}_{1,2}|2\right\rangle = \left\langle 2\left|\not{p}_4\right|1\right]\langle 12\rangle, \quad \text{(H.449)}$$

which gives the following compact form for the amplitude,

$$iM_4(1_\gamma^+, 2_\gamma^-, 3_\phi, 4_\phi) = \frac{\kappa^2}{4}\frac{\left\langle 2\left|\not{p}_4\right|1\right]^2}{\langle 12\rangle[12]} \quad \text{(H.450)}$$

3. Compute $\left|M_4(1_\gamma^+, 2_\gamma^-, 3_\phi, 4_\phi)\right|^2$.

Solution. Using $\left\langle 2\left|\not{p}_4\right|1\right]^\dagger = \left\langle 1\left|\not{p}_4\right|2\right]$, the squared modulus of the amplitude is

$$\left|M_4(1_\gamma^+, 2_\gamma^-, 3_\phi, 4_\phi)\right|^2 = \frac{\kappa^4}{16}\frac{\left\langle 2\left|\not{p}_4\right|1\right]^2\left\langle 1\left|\not{p}_4\right|2\right]^2}{\langle 12\rangle^2[12]^2}.$$

Note that

$$\begin{aligned}
\left\langle 2\left|\not{p}_4\right|1\right]\left\langle 1\left|\not{p}_4\right|2\right] &= \lambda^a(2)(p_4\cdot\bar{\sigma})_{a\dot{b}}\tilde{\lambda}^{\dot{b}}(1)\lambda^c(1)(p_4\cdot\bar{\sigma})_{c\dot{d}}\tilde{\lambda}^{\dot{d}}(2) \\
&= \operatorname{tr}\left(\not{p}_4 p_1\cdot\sigma\not{p}_4 p_2\cdot\sigma\right) = p_{4\mu}p_{1v}p_{4\rho}p_{2\delta}\operatorname{tr}\left(\bar{\sigma}^\mu\sigma^v\bar{\sigma}_\sigma\sigma^\delta\right) \\
&= 2p_{4\mu}p_{1v}p_{4\rho}p_{2\sigma}\left(\eta^{\mu v}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{v\sigma} + \eta^{\mu\sigma}\eta^{v\rho} + i\epsilon^{\mu v\rho\sigma}\right) \\
&= 2\left(2\left(p_1\cdot p_4\right)\left(p_2\cdot p_4\right) - p_4^2\left(p_1\cdot p_2\right)\right) \\
&= s_{13}s_{14} - m^4,
\end{aligned} \quad \text{(H.451)}$$

[2]Whether it is q^μ of the shift or of the polarization vector is understood from the context.

where we have introduced the Lorentz invariants $s_{ij} \equiv (p_i + p_j)^2$ and used $s_{24} = s_{13}$ and

$$s_{12} + s_{13} + s_{14} = 2m^2$$

(both follow from momentum conservation). Therefore, the squared amplitude reads

$$\left| M_4(1^+_\gamma, 2^-_\gamma, 3_\phi, 4_\phi) \right|^2 = \frac{\kappa^4}{16} \frac{(s_{13}s_{14} - m^4)^2}{s_{12}^2}. \tag{H.452}$$

The differential cross-section with respect to the solid angle of the outgoing photon is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 \, \mathrm{s}_{14}} \left| M_4(1^+_\gamma, 2^-_\gamma, 3_\phi, 4_\phi) \right|^2 . \tag{H.453}$$

Let us now consider the limit of long wavelength photons, namely $\omega = |p_{1,2}| \ll m$. In this limit, the Lorentz invariants that appear in the cross-section simplify into

$$\begin{aligned} s_{12} &\approx 4\omega^2 \sin^2 \frac{\theta}{2} \\ s_{13} &\approx m^2 - 2m\omega - 4\omega^2 \sin^2 \frac{\theta}{2}, \\ s_{14} &\approx m^2 + 2m\omega, \end{aligned}$$

where ω is the photon energy and θ its deflection angle in the center-of-mass frame (which is also the frame of the massive scalar particle in this limit). For large enough impact parameters, the deflection angle is small, $\theta \ll 1$. Thus, we obtain in this limit

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \frac{16\mathrm{G}_\mathrm{N}^2 \, \mathrm{m}^2}{\theta^4}. \tag{H.454}$$

In order to determine the deflection angle as a function of the impact parameter b, consider a flux $\mathcal{F}$ of photons along the z direction, with the massive scalar at rest at the origin. Within this flux, consider specifically the incoming photons in a ring of radius b and width db. The number of photons flowing per unit time through this ring is

$$2\pi b \mathcal{F} db. \tag{H.455}$$

All these photons are scattered in the range of polar angles $[\theta(b) + d\theta, \theta(b)]$ (note that $d\theta$ is negative for db > 0, because the deflection angle decreases at largerb), which corresponds to the solid angle,

$$d\Omega = -2\pi \sin(\theta(b)) d\theta. \tag{H.456}$$

By definition, the number of scattering events is the flux times the cross-section, i.e.,

$$2\pi b \mathcal{F} db = \mathcal{F} \frac{d\sigma}{d\Omega} d\Omega, \tag{H.457}$$

which can be integrated for small angles into

$$\theta(b) = \frac{4G_N m}{b}\,, \tag{H.458}$$

where the integration constant has been chosen so that the deflection vanishes when $b \to \infty$. This is indeed the standard formula from general relativity, which can be derived by considering geodesics in the Schwarzschild metric.

H.38 Scattering of gravitational waves by a mass

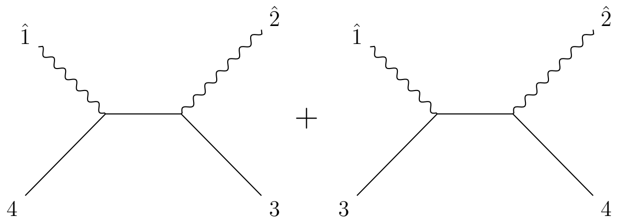
Consider the amplitude for the scattering of a graviton off a massive scalar, $M_4(1_{\rm h}^-, 2_{\rm h}^+, 3_\phi, 4_\phi)$.

1. Consider the shift

$$\left|\hat{2}\right] = |2]\,, \quad |\hat{2}\rangle = |2\rangle - z|1\rangle, \quad \left|\hat{1}\right] = |1] + z\,|2]\,, \quad |\hat{1}\rangle = |1\rangle. \tag{H.459}$$

Assume $M_4(z) \to 0$ as $z \to \infty$ and write the BCFW recursion relation.

Solution. With this shift, M_4 has two contributing diagrams in the BCFW recursion relation,



(H.460)

$$\begin{aligned} iM_4(1_{\rm h}^-, 2_{\rm h}^+, 3_\phi, 4_\phi) =& iM_3\left(\hat{1}_{\rm h}^-, 4_\phi, \widehat{p}_{23,\phi}\right) \frac{i}{s_{23} - m^2} iM_3\left(\hat{2}_{\rm h}^+, 3_\phi, -\widehat{p}_{23,\phi}\right) \\ &+ iM_3\left(\hat{1}_{\rm h}^-, 3_\phi, \widehat{p}_{24,\phi}\right) \frac{i}{s_{24} - m^2} iM_3\left(\hat{2}_{\rm h}^+, 4_\phi, -\widehat{p}_{24,\phi}\right)\,. \end{aligned} \tag{H.461}$$

2. Compute the amplitude setting p_2 as reference to the polarisation vector of graviton 1 and p_1 as reference to the polarisation vector of graviton 2.

Solution. We use the expressions of $M_3(\phi, \phi, {\rm h})$ found in eqs. (H.432) and (H.433),

$$\begin{aligned} iM_4(1_{\rm h}^-, 2_{\rm h}^+, 3_\phi, 4_\phi) &= -i\frac{\kappa^2}{4} \frac{\langle\hat{1}\big|\not{p}_4\,|2]^2 \,\langle 1|\not{p}_3\big|\hat{2}\big]^2}{(s_{23} - m^2)[\hat{1}2]^2\langle 1\hat{2}\rangle^2} - i\frac{\kappa^2}{4} \frac{\langle\hat{1}\big|\not{p}_3\,|2]^2 \,\langle 1|\not{p}_4\big|\hat{2}\big]^2}{(s_{24} - m^2)[\hat{1}2]^2\langle 1\hat{2}\rangle^2} \\ &= -i\frac{\kappa^2}{4} \frac{\langle 1|\not{p}_3\,|2]^4}{[12]^2\langle 12\rangle^2} \left\{ \frac{1}{(s_{23} - m^2)} + \frac{1}{(s_{24} - m^2)} \right\} \\ &= i\frac{\kappa^2}{16} \frac{\langle 1|\not{p}_3\,|2]^4}{[12]\langle 12\rangle} \frac{1}{(p_2 \cdot p_3)(p_2 \cdot p_4)} \end{aligned} \tag{H.462}$$

3. Compute $\left|M_4(1_{\rm h}^-, 2_{\rm h}^+, 3_\phi, 4_\phi)\right|^2$.

   Solution. We find

$$\left|M_4(1_{\rm h}^-, 2_{\rm h}^+, 3_\phi, 4_\phi)\right|^2 = \left|M_4(1_\gamma^-, 2_\gamma^+, 3_\phi, 4_\phi)\right|^2 \left\{1 - \frac{m^2 s_{12}}{(s_{13}-m^2)\,(s_{14}-m^2)}\right\} \tag{H.463}$$

   In the limit of a graviton of small energy (i.e., a gravitational wave of long wavelength) and small deflection angle (i.e., at large impact parameter), the second factor on the right-hand side becomes equal to 1 , and we have

$$\left|M_4(1_{\rm h}^-, 2_{\rm h}^+, 3_\phi, 4_\phi)\right|^2 \overset{\omega \ll m}{\approx} \left|M_4(1_\gamma^-, 2_\gamma^+, 3_\phi, 4_\phi)\right|^2 \tag{H.464}$$

   This implies that in this limit the bending of a gravitational wave by a mass is the same as the bending of a light ray (but there are some differences beyond this limit).

H.39 Solution of the Dirac equation

Whenever an explicit representation is asked we consider the chiral representation of the gamma matrices, which is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \qquad P_{R,L} := \frac{1 \pm \gamma^5}{2}\,, \tag{H.465}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{H.466}$$

where $\sigma^\mu := (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu := (\mathbb{1}, -\vec{\sigma})$. The generators of the Lorentz transformations for spinors are given by

$$(\mathcal{S}^{\mu\nu})_{\alpha\beta} = \frac{i}{4}([\gamma^\mu, \gamma^\nu])_{\alpha\beta}, \qquad \Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{S}^{\mu\nu}\right). \tag{H.467}$$

1. Consider the Dirac equation for positive frequencies solution, $\psi(x) := u(p)e^{-ip\cdot x}$. Show that the solution for a particle at rest with normalization $\bar{u}\,u = 2m$, can be written as:

$$u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \qquad (p_0)^\mu = (m, 0, 0, 0)\,. \tag{H.468}$$

   where ξ are two-component spinors satisfying $\xi^\dagger \xi = 1$

   Solution. The equation that we need to solve is $(\gamma^\mu p_\mu - m)u(p) = 0$ for $p^\mu = (m, 0, 0, 0)$, then it reduces to

$$(\gamma^0 - 1)u(p) = \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p) = 0\,. \tag{H.469}$$

The only non trivial solution of this expression is given by $u(p) \propto \begin{pmatrix} \xi \\ \xi \end{pmatrix}$.

2. Split the component of the spinor into the the chiral components of $u(p)_{R,L} := P_{R,L}u(p)$ as $u(p) = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}$ and show that they do not mix under a boost.

Solution. We have for boosts,

$$-\frac{i}{2}[(\omega_{0i}\mathcal{S}^{0i}) + (\omega_{i0}\mathcal{S}^{i0})] = -\frac{i}{2}\Big[- i\omega_{0i}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\Big], \tag{H.470}$$

so with $w_{0i} = \eta_i$

$$\Lambda_{1/2} = \exp\Big(\frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\Big) = \begin{pmatrix} \exp\big(\frac{1}{2}\vec{\sigma}\vec{\eta}\big) & 0 \\ 0 & \exp\big(-\frac{1}{2}\vec{\sigma}\vec{\eta}\big) \end{pmatrix}, \tag{H.471}$$

we see that they do not mix.

3. Boost the solution of the rest system with the velocity $\vec{v} = \dfrac{\vec{p}}{E}$ to obtain the general solution and express the result in terms of E, $|\vec{p}|$ and m. Use the parametrisation,

$$p^\mu = \begin{pmatrix} E \\ |\vec{p}|\cos(\phi)\sin(\theta) \\ |\vec{p}|\sin(\phi)\sin(\theta) \\ |\vec{p}|\cos(\theta) \end{pmatrix}. \tag{H.472}$$

Hint. The rapidity is $\vec{\eta} = \eta\dfrac{\vec{p}}{|\vec{p}|} = \operatorname{atanh}\big(\dfrac{|\vec{p}|}{E}\big)\dfrac{\vec{p}}{|\vec{p}|}$.

Hint. Use

$$\begin{aligned} \cosh\Big(\frac{\eta}{2}\Big) &= \sqrt{\frac{1}{2}\left(\cosh(\eta) + 1\right)}, \\ \sinh\Big(\frac{\eta}{2}\Big) &= \sqrt{\frac{1}{2}\left(\cosh(\eta) - 1\right)}, \\ \cosh(\operatorname{atanh}(x)) &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned} \tag{H.473}$$

Solution. To perform the matrix exponential we use,

$$\begin{aligned} (\vec{\sigma}\vec{\eta})^2 &= \eta^2 \underbrace{\sigma_i\sigma_j}_{\mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k} \frac{p_ip_j}{|\vec{p}|^2} \\ &= \mathbb{1}\eta^2\delta_{ij}\frac{p_ip_j}{|\vec{p}|^2} \\ &= \mathbb{1}\eta^2. \end{aligned} \tag{H.474}$$

So

$$\exp\Big(\frac{1}{2}\vec{\sigma}\vec{\eta}\Big) = \mathbb{1}\sum_n \frac{1}{(2n)!}\left(\frac{1}{2}\eta\right)^{2n} + \frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\sum_n \frac{1}{(2n+1)!}\left(\frac{1}{2}\eta\right)^{2n+1}$$
$$= \cosh\Big(\frac{\eta}{2}\Big)\mathbb{1} + \sinh\Big(\frac{\eta}{2}\Big)\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\,. \tag{H.475}$$

where

$$\cosh\Big(\frac{\eta}{2}\Big) = \sqrt{\frac{1}{2}\left(\cosh(\eta)+1\right)} = \sqrt{\frac{E+m}{2m}}\,,$$
$$\sinh\Big(\frac{\eta}{2}\Big) = \sqrt{\frac{1}{2}\left(\cosh(\eta)-1\right)} = \sqrt{\frac{E-m}{2m}}\,. \tag{H.476}$$

H.40 Irreducible representations of SU(2)

We consider the fully symmetric SU(2) tensor,

$$\Psi_{(i_1,\ldots,i_{2j})}\,, \tag{H.477}$$

where i is the SU(2) index by taking the direct product of spin-1/2 wave-functions (we will make things more concrete below). We have the action of the rotation generators,

$$(\vec{J}\Psi)_{(i_1,\ldots,i_{2j})} = \frac{1}{2}(\vec{\sigma})_{i_1}^{j_1}\Psi_{(j_1,\ldots,i_{2j})} + \cdots + \frac{1}{2}(\vec{\sigma})_{i_{2j}}^{j_{2j}}\Psi_{(i_1,\ldots,j_{2j})}\,. \tag{H.478}$$

- How many components does that tensor have?

  Solution. That is the purely combinatorical question of asking in how many ways we can put r indistinguishable objects (the indices) into d boxes (the values of the indices). The answer is

$$\begin{pmatrix} d+r-1 \\ r \end{pmatrix} = \frac{(d+r-1)!}{r!\,(d-1)!}\,. \tag{H.479}$$

  and for $d=2$ (up,down) and $r=2j$ we have $(1+2j)!\,/(2j)! = 1+2j$ possibilities.

- Consider now $i \in \{\uparrow,\downarrow\}$ and $j=1$ and give all components explicitly.

  Solution. We have the three components

$$T_{(a,b)} = \begin{cases} |\uparrow\uparrow\rangle & (m=1) \\ \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) & (m=0) \\ |\downarrow\downarrow\rangle & (m=-1) \end{cases} \tag{H.480}$$

  with the notation $|\downarrow\downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle$.

- Compute $(\vec{J}^2\Psi)_{(i_1,\dots,i_{2j})}$.

  Solution. We have to consider

$$(\vec{J}\underbrace{\otimes\mathbb{1}\otimes\dots\otimes\mathbb{1}}_{\text{(2j-1) times}}+\dots\mathbb{1}\otimes\dots\otimes\vec{J})^2\Psi\,, \tag{H.481}$$

  So we have $2j$-terms of the form $\mathbb{1}\otimes\dots\otimes\vec{J}^2\otimes\dots\otimes\mathbb{1}$ with

$$\frac{1}{4}(\vec{\sigma})^l_{i_j}(\vec{\sigma})^{j_j}_l\Psi_{(i_1,\dots j_j\dots,i_{2j})}=\frac{1}{4}(2\delta^l_l\delta^{j_j}_{i_j}-\delta^l_{i_j}\delta^{j_j}_l)\Psi_{(i_1,\dots j_j\dots,i_{2j})}=\frac{3}{4}\Psi_{(i_1,\dots i_j\dots,i_{2j})}\,, \tag{H.482}$$

  and $(2j)^2-2j$ mixed terms of the form,

$$\frac{1}{4}(\vec{\sigma})^{j_k}_{i_k}(\vec{\sigma})^{j_l}_{i_l}\Psi_{(i_1,\dots j_k\dots j_l\dots,i_{2j})}=\frac{1}{4}(2\delta^{j_l}_{i_k}\delta^{j_k}_{i_l}-\delta^{j_k}_{i_k}\delta^{j_l}_{i_l})\Psi_{(i_1,\dots j_k\dots j_l\dots,i_{2j})}=\frac{1}{4}\Psi_{(i_1,\dots i_k\dots i_l\dots,i_{2j})}\,, \tag{H.483}$$

  where we used the symmetry of the tensor. So we have

$$(\vec{J}^2\Psi)_{(i_1,\dots,i_{2j})}=(2j\frac{3}{4}+2j(2j-1)\frac{1}{4})\Psi_{(i_1,\dots,i_{2j})}=j(1+j)\Psi_{(i_1\dots,i_{2j})}\,, \tag{H.484}$$

  as expected.

We now want to turn to the irreducible representation of $SU(2)$ and see that the fully symmetric $SU(2)$-tensor defined above is a representation[3]. $g\in SU(2)$ can be written as

$$g=\begin{pmatrix} a & b\\ -b^* & a^*\end{pmatrix}, \qquad a,b\in\mathbb{C}\,, \tag{H.485}$$

with $|a|^2+|b|^2=1$. We denote $\{e_1,e_2\}$ as the basis of the complex two-dimensional space $\mathbb{C}^2$ on which g acts with

$$g(e_i)=g^j_ie_i\,. \tag{H.486}$$

[3]A representation is the map $\rho:SU(2)\to GL(V)$ where V is a vector space called representation space. If there is no ambiguity V is referred to as representation.

A basis of the $(2j+1)$-dimensional complex vector space $\mathbb{V}^{(j)}$ is

$$\begin{aligned}
\{e_0^{(j)} &= \sqrt{(2j)!}S_{2j}(e_1^{2j}) = \sqrt{(2j)!}S_{2j}(\overbrace{e_1 \otimes \ldots \otimes e_1}^{2j \text{ times}})\,, \\
e_1^{(j)} &= \sqrt{(2j)!}S_{2j}(e_1^{2j-1} \otimes e_2) = \sqrt{(2j)!}S_{2j}(\overbrace{e_1 \otimes \ldots \otimes e_1}^{2j-1 \text{ times}} \otimes e_2)\,, \\
&\vdots \\
e_k^{(j)} &= \sqrt{(2j)!}S_{2j}(e_1^{2j-k} \otimes e_2^k)\,, \\
&\vdots \\
e_{2j}^{(j)} &= \sqrt{(2j)!}S_{2j}(e_2^{2j})\}\,,
\end{aligned} \tag{H.487}$$

where S_{2j} is the symmetrisation map. We can regard the basis vectors as homogeneous polynomials of degree $2j$ in e_1 and e_2 with e.g.

$$e_1^{(2)} = e_1^3 e_2\,. \tag{H.488}$$

1. What is the range of k in eq. (H.487).

   Solution. $\quad k = 0, \ldots, 2j$.

2. Compute the action of $SU(2)$ on $\mathbb{V}^j$ by evaluating $g(e_k^{(j)})$.

   Solution. We have

$$\begin{aligned}
g(e_k^{(j)}) = (g(e_1))^{2j-k}(g(e_2))^k &= (ae_1 + be_2)^{2j-k}(-\bar{b}e_1 + \bar{a}e_2)^k \\
&= \sum_{m=0}^{2j-k} \binom{2j-k}{m} (ae_1)^{2j-k-m}(be_2)^m \sum_{l=0}^{k} \binom{k}{l} (-\bar{b}e_1)^{k-l}(\bar{a}e_2)^l \\
&= \sum_{m=0}^{2j-k} \sum_{l=0}^{k} \binom{2j-k}{m} \binom{k}{l} (a)^{2j-k-m}(b)^m(-\bar{b})^{k-l}(\bar{a})^l \underbrace{e_1^{2j-(m+l)} e_2^{m+l}}_{e_{m+l}^{(j)}}\,.
\end{aligned} \tag{H.489}$$

   If we relabel $m + l = p$, in the double sum above p takes values from $p = 0, \ldots, 2j$ and we can write

$$g(e_k^{(j)}) = (\mathcal{D}^{(j)})_k^p e_p^{(j)}\,, \tag{H.490}$$

   so we have a (irreducible) representation of $\rho : SU(2) \to GL(\mathbb{V}^j)$ with $\mathcal{D}^{(j)} \in GL(\mathbb{V}^j)$.

3. We now define the normalized spinors as the basis of $\mathbb{V}^j$,

$$|jm\rangle = \epsilon_m^{(j)} = \frac{e_1^{j+m} e_2^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}\,, \tag{H.491}$$

where $e_1^{j+m}e_2^{j-m}$ denotes the symmetrized product and $m = j - k$. What is the range of m?

Solution. Since $k = 0, 1, \ldots, 2j$ we have $m = -j, , -j+1, \ldots, j$, which we are more used to in physics.

4. Consider now the case, where the basis is not chosen as eigenstates of J_z, but as an Eigenstate of $J_{\vec{n}}$. How would you go about relating the symmetric tensors $|jm_z\rangle$ and $|jm_{\vec{n}}\rangle$? Do not do the computation, just outline the approach.

   Solution. We would either find the eigenvectors of $\vec{n}\vec{S}$, which are the eigenstates $|jm_{\vec{n}}\rangle$ or alternatively obtain them by rotating e_1 and e_2 with

$$e_i' = \exp\left(-i\frac{\phi}{2}\sigma_2\right)\exp\left(-i\frac{\psi}{2}\sigma_3\right)e_i\,. \tag{H.492}$$

   So the eigenstates $|jm_{\vec{n}}\rangle$ and $|jm_z\rangle$ are simply related by a $SU(2)$ transformation and with our previous computation (H.490) we can immediately obtain the transformed symmetric spinor by plugging in $a, \bar{a}, b, \bar{b}$.

We now want to demonstrate unitarity. We consider the standard basis of $\mathbb{C}^2$ and $u = (u^1, u^2) \in \mathbb{C}^2$ with $u = u^1 e_1 + u^2 e_2$. We have the usual inner product,

$$(u, v) = u^1\bar{v^1} + u^2\bar{v^2}\,. \tag{H.493}$$

The vector $1/\sqrt{(2j)!}\overbrace{u\otimes\cdot\otimes u}^{2j\text{ times}}$ in the tensor space $\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2$ then projects on $u^{(j)} \in \mathbb{V}^{(j)}$ with components along $\epsilon_i^{(j)}$,

$$(u^{(j)})^0 = \frac{(u^1)^{2j}}{\sqrt{(2j)!}},\ (u^{(j)})^1 = \frac{(u^1)^{2j-1}(u^2)}{\sqrt{(2j-1)!}},\ \ldots\ ,(u^j)^{(2j)} = \frac{(u^2)^{2j}}{\sqrt{(2j)!}}\ , \tag{H.494}$$

and the induced inner product,

$$(u^{(j)}, v^{(j)}) = \sum_{k=0}^{2j}(u^{(j)})^k(\bar{v}^{(j)})^k\,, \tag{H.495}$$

on $\mathbb{V}^j$. Under $g \in SU(2)$, $u^{(j)}$ transforms as

$$\begin{aligned}(u^{(j)})^k \to ((u^{(j)})')^k &= \frac{1}{\sqrt{(2j-k)!\,k!}}((u')^1)^{2j-k}((u')^2)^k \\ &= \frac{1}{\sqrt{(2j-k)!\,k!}}(au^1 - \bar{b}u^2)^{2j-k}(bu^1 + \bar{a}u^2)^k\,.\end{aligned} \tag{H.496}$$

1. Show $((u^{(j)})', (v^{(j)})') = (u^{(j)}, v^{(j)})$: the (2j+1)-dimensional representation of SU(2) on $\mathbb{V}^j$, specified by the basis (H.487) is unitary.

Solution. We consider

$$\begin{aligned}
(2j)!\left((u^{(j)})', (v^{(j)})'\right) &= \sum_{k=0}^{2j} ((u^{(j)})')^k \overline{((\bar{v}^{(j)})')}^k \\
&= (2j)! \sum_{k=0}^{2j} \frac{1}{\sqrt{(2j-k)!\,k!}} ((u')^1)^{2j-k}((u')^2)^k \frac{1}{\sqrt{(2j-k)!\,k!}} (\overline{(v')^1})^{2j-k}(\overline{(v')^2})^k \\
&= \sum_{k=0}^{2j} \binom{2j}{k} \left((u')^1\overline{((v')^1)}\right)^{2j-k} \left((u')^2\overline{((v')^2)}\right)^k \\
&= \left((u')^1\overline{(v')^1} + (u')^2\overline{(v')^2}\right)^{2j} \\
\text{(unitarity of g)} &= \left((u)^1\overline{(v)^1} + (u)^2\overline{(v)^2}\right)^{2j} \\
&= (2j)!\left((u^{(j)}), (v^{(j)})\right). \qquad \text{(H.497)}
\end{aligned}$$

So the irreducible $(2j+1)$-dimensional representation of $SU(2)$ on $\mathbb{V}^{(j)}$ is indeed unitary.

Consider now

$$((u^j)')^m = \left(D^{(j)}\right)^m_{m'} (u^j)^{m'}\,, \qquad \text{(H.498)}$$

where again $m = j - k$ such that

$$(u^j)^m = \frac{1}{\sqrt{(j+m)!\,(j-m)!}} (u^1)^{j+m}(u^2)^{j-m}\,. \qquad \text{(H.499)}$$

1. Compute $\left(D^{(j)}\right)^m_{m'}$. What is the representation space?

   Solution. The vector with the components $(u^j)^m$ is a vector in the linear space of homogeneous polynomial of degree $2j$ in $\mathbb{C}^2$ (see the definition of $(u^j)^m$). The computation of the group action is

$$\begin{aligned}
((u^j)')^m &= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} ((u^1)')^{j+m}((u^2)')^{j-m} \\
&= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} (au^1 - \bar{b}u^2)^{j+m}((bu^1 - \bar{a}u^2))^{j-m} \\
&= \frac{1}{\sqrt{(j+m)!\,(j-m)!}} \sum_{p=0}^{j+m} \binom{j+m}{p} (au^1)^{j+m+p}(-\bar{b}u^2)^p \sum_{q=0}^{j-m} \binom{j-m}{q} (bu^1)^{j-m-q}(-\bar{a}u^2)^q \\
&= \sum_{p=0}^{j+m}\sum_{q=0}^{j-m} \frac{\sqrt{(j+m)!\,(j-m)!}}{(p)!\,(j+m-p)!\,q!\,(j-m-q)!} (-1)^p a^{j+m-p}(\bar{b})^p b^{j-m-q}(\bar{a})^q (u^1)^{2j-(p+q)}(u^2)^{(p+q)}\,.
\end{aligned} \qquad \text{(H.500)}$$

   However, we want to have an action which is

$$((u^j)')^m = \left(D^{(j)}\right)^m_{m'} (u^j)^{m'} = \left(D^{(j)}\right)^m_{m'} \left(\frac{1}{\sqrt{(j+m')!\,(j-m')!}} (u^1)^{j+m'}(u^2)^{j-m'}\right), \qquad \text{(H.501)}$$

so we need to find a index shift in (H.500), such that the exponents are $(j+m')$, $(j-m')$. The obvious choice is $m' = j-(p+q)$, where m' ranges over the $(2j+1)$ values $j, j-1, \ldots, -j$. So we find

$$((u^j)')^m = \sum_{p=0}^{j+m} \sum_{q=0}^{j-m} \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{p!\,(j+m-p)!\,(j-m'-p)!\,(p+m'-m)!}(-1)^p \\ \times a^{j+m-p}(\bar{b})^p b^{j-m-q}(\bar{a})^q \Big(\frac{1}{\sqrt{(j+m')!\,(j-m')!}} (u^1)^{j+m'} (u^2)^{j-m'} \Big) . \tag{H.502}$$

This representation however is still not optimal, since we would need to find for a given m' all the p,q-combinations which fulfil $m' = j-(p+q)$. Therefore let us rewrite the double sum $\sum_{p=0}^{j+m}\sum_{q=0}^{j-m}$ as a sum in m': $\sum_{m'=-j}^{j}\sum_{p=\tilde{p}_{min}}^{\tilde{p}_{max}}$. To get the summation bounds we look at $m' = j-(p+q)$ or equivalently $p = j-m'-q$. At the maximum value of q, $q_{max} = j-m$, we have $p = m-m'$. However, if $m-m' < 0$, this is not a allowed value of p, since $p \geq 0$. So we have the lower summation bound of p is $\max(0, m-m')$. To find the upper summation bound, we look at the minimum value of q, $q_{min} = 0$, for which $p = j-m'$. So the upper summation bound of $p = \min(j-m', j+m)$, where the min is taken to prevent the case $j-m' > j+m$, which is not within the summation range of p. This means our transformation is

$$((u^j)')^m = \sum_{m'=-j}^{j} \sum_{p=\max(0,m-m')}^{\min(j-m',j+m)} \frac{(-1)^p\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{p!\,(j+m-p)!\,(j-m'-p)!\,(p+m'-m)!} \\ \times a^{j+m-p}(\bar{a})^{j-m'-p}(\bar{b})^p b^{p+m'-m}(u^j)^{m'} , \tag{H.503}$$

which tells us that

$$\left(D^{(j)}\right)^m_{m'} = \sum_{p=\max(0,m-m')}^{\min(j-m',j+m)} \frac{(-1)^p\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{p!\,(j+m-p)!\,(j-m'-p)!\,(p+m'-m)!} a^{j+m-p}(\bar{a})^{j-m'-p}(\bar{b})^p b^{p+m'-m} , \tag{H.504}$$

are the matrix elements of the unitary representation.

2. Evaluate $D^{(1/2)}$ and $D^{(1)}$. Are these matrices unitary as expected?

Solution. We can simply apply our formula (H.504) to the case $j = 1/2$,

$$(D^{(\frac{1}{2})})^{\frac{1}{2}}_{\frac{1}{2}} = a, \qquad (D^{(\frac{1}{2})})^{-\frac{1}{2}}_{\frac{1}{2}} = b, \qquad (D^{(\frac{1}{2})})^{\frac{1}{2}}_{-\frac{1}{2}} = -\bar{b}, \qquad (D^{(\frac{1}{2})})^{-\frac{1}{2}}_{-\frac{1}{2}} = \bar{a} , \tag{H.505}$$

and we find

$$D^{(\frac{1}{2})} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = g , \tag{H.506}$$

as expected.

For the case $j = 1$ we find

$$D^{(1)} = \begin{pmatrix} (D^{(1)})_1^1 & (D^{(1)})_1^0 & (D^{(1)})_1^{-1} \\ (D^{(1)})_0^1 & (D^{(1)})_0^0 & (D^{(1)})_0^{-1} \\ (D^{(1)})_{-1}^1 & (D^{(0)})_{-1}^0 & (D^{(1)})_{-1}^{-1} \end{pmatrix} = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ -\sqrt{2}a\bar{b} & |a|^2-|b|^2 & \sqrt{2}\bar{a}b \\ \bar{b}^2 & -\sqrt{2}\bar{a}\bar{b} & \bar{a}^2 \end{pmatrix}. \tag{H.507}$$

and unitarity as well as $\det\big(D^{(1)}\big) = 1$ can easily be verified. (see e.g. notebook enclosed in the Tutorials.)

3. For the rotation around the x-axis $R_2 = \exp\Big(-i\frac{\theta}{2}\sigma_2\Big)$ one has $(D^{(\ell)})_0^m(\theta) = \sqrt{\frac{(\ell+|m|)!}{(\ell-|m|)!}}\,P_\ell^m(\cos\theta)$, where $P_\ell^m(\cos\theta)$ is the associated Legendre polynomial used for spherical harmonics. Verify it for the case $j = 1$. What happens for a rotation around the z-axis?

Solution. We compute

$$R_2 = \exp\Big(-i\frac{\theta}{2}\sigma_2\Big) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \tag{H.508}$$

so

$$a = \cos\left(\frac{\theta}{2}\right), \qquad b = -\sin\left(\frac{\theta}{2}\right), \tag{H.509}$$

and

$$\begin{aligned} (D^{(1)})_0^1 &= \sqrt{2}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{\sqrt{2}}, \\ (D^{(1)})_0^0 &= \sin^2\left(\frac{\theta}{2}\right) - \cos^2\left(\frac{\theta}{2}\right) = \cos(\theta), \\ (D^{(1)})_0^{-1} &= -\sqrt{2}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \cos(\theta) = -\frac{\sin(\theta)}{\sqrt{2}}. \end{aligned} \tag{H.510}$$

The differential equation for the associated Legendre polynomials is

$$\frac{\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f(\theta)}{\partial\theta}\right)}{f(\theta)} + j(j+1)\sin^2(\theta) - m^2 = 0, \tag{H.511}$$

and we check the claim by plugging in our solutions,

$$
\begin{aligned}
j = 1, m = 1, \qquad & \frac{\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\sin(\theta)}{\partial\theta}\right)}{\sin(\theta)} + (1+1)\sin^2(\theta) - 1^2 \\
& = \sin^2(\theta) + \cos^2(\theta) - 1 = 0\,, \\
j = 1, m = 0 \qquad & \frac{\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\cos(\theta)}{\partial\theta}\right)}{\cos(\theta)} + (1+1)\sin^2(\theta) - 0^2 = 0\,, \\
j = 1, m = -1 \qquad & -\frac{\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial(-\sin(\theta))}{\partial\theta}\right)}{\sin(\theta)} + (1+1)\sin^2(\theta) - (-1)^2 \\
& = \sin^2(\theta) + \cos^2(\theta) - 1 = 0\,.
\end{aligned}
\tag{H.512}
$$

For a rotation around the z-axis with

$$
R_3 = \exp\left(-i\frac{\phi}{2}\sigma_3\right) = \begin{pmatrix} e^{-\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix}\,,
\tag{H.513}
$$

we have

$$
a = e^{-\frac{i\phi}{2}}\,, \qquad\qquad b = 0\,,
\tag{H.514}
$$

and one will get

$$
(D^{(j)(R_3)})^m_{m'} = \delta_{mm'}\exp(-im\phi)\,,
\tag{H.515}
$$

which means we can generate the spherical harmonics.

H.41 $q\bar{q} \to gg$ with massive quarks

Consider the process in fig. H.16 for a massive quark-antiquark pair and two opposite helicity gluons. Recall that for massive particles the property of being a helicity eigenstate is frame-dependent. We have extended the formalism derived for massless particles in the following way $p_{a\dot{a}} = (p\cdot\bar{\sigma})_{a\dot{a}} = \lambda^I_a(p)\tilde{\lambda}_{\dot{a},I}(p)$ with I index for the representation of $SU(2)$ of spin 1/2.

1. Compute the amplitude using the massive spinor-helicity formalism.
2. Verify that in the massless limit you obtain the Parke-Taylor formula for $q\bar{q} \to gg$.

Hint *As a first step, write the amplitude with the Feynman rules passing by the Dirac spinors of the form*

$$
u^I(p) = \begin{pmatrix} \tilde{\lambda}^{\dot{a},I} \\ \lambda^I_a \end{pmatrix}\,.
\tag{H.516}
$$

The amplitude will be $M(1^I, 2^J, 3^-, 4^+) \sim \bar{u}^I(p_1)\cdots v^J(p_2)$.

Hint *Choose the polarisation vectors as* $\epsilon^*(p_3,p_4)$, $\epsilon^*(p_4,p_3)$.

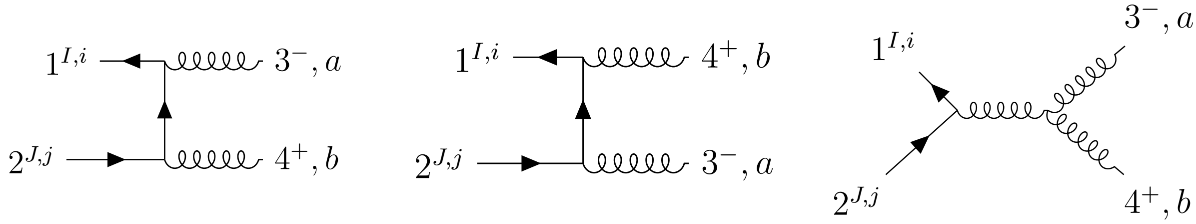


Figure H.16: Relevant diagrams for $q\bar{q} \to gg$.

Solution. For brevity in the following we denote ϵ^* as ϵ. We denote by i,j,k the colour indices in the fundamental representation, a,b the colour indices in the adjoint representation and I,J the little group indices for the spin 1/2 representation of SU(2). We can use the results of sec. H.12 and write the amplitude as

$$\begin{aligned} iM_4(1_q^I, 2_{\bar{q}}^J, 3, 4) = \left(\frac{ig}{\sqrt{2}}\right)^2 \bar{u}_i^I(p_1) \Big\{ &(T^{a_3}T^{a_4})^i_j \left(\not\epsilon(p_3)\frac{i(\not p_1 + \not p_3 + m)}{s_{13}-m^2}\not\epsilon(p_4) \right. \\ &\left. + \frac{-i}{s_{12}}\left[2\not\epsilon_4(p_4\cdot\epsilon_3) - 2\not\epsilon_3(p_3\cdot\epsilon_4) + (\not p_3 - \not p_4)(\epsilon_3\cdot\epsilon_4)\right]\right) \\ &+ (T^{a_4}T^{a_3})^i_j \left(\not\epsilon(p_4)\frac{i(\not p_1 + \not p_4 + m)}{s_{14}-m^2}\not\epsilon(p_3) \right. \\ &\left. + \frac{i}{s_{12}}\left[2\not\epsilon_4(p_4\cdot\epsilon_3) - 2\not\epsilon_3(p_3\cdot\epsilon_4) + (\not p_3 - \not p_4)(\epsilon_3\cdot\epsilon_4)\right]\right)\Big\} v^{J,j}(p_2)\,. \end{aligned} \tag{H.517}$$

The same reasoning about colour ordering holds also here (the part proportional to $(T^{a_3}T^{a_4})$ is equal to the part proportional to $(T^{a_4}T^{a_3})$ under the swap of 3 with 4) and we only need to calculate one colour-stripped amplitude, that we choose to be $A_4(1^I, 2^J, 3^-, 4^+)$.. We set the polarisation vectors to $\epsilon(p_3,p_4)$, $\epsilon(p_4,p_3)$ and the amplitude becomes

$$i\,A_4(1_q^I, 2_{\bar{q}}^J, 3_g^-, 4_g^+) = \left(\frac{i}{\sqrt{2}}\right)^2 \bar{u}^I(p_1)\left(\not\epsilon^-(3,4)\frac{i(\not p_1 + \not p_3 + m)}{s_{13}-m^2}\not\epsilon^+(4,3)\right)v^J(p_2). \tag{H.518}$$

Since the two-components Weyl spinors (at fixed I) of u^I live in different spaces, in bra-ket notation $u^I(p) \sim \left|p^I\right\rangle + |p^I]$. Moreover, we use

$$\gamma\cdot\epsilon^+(p,k) = \frac{\sqrt{2}}{\langle kp\rangle}\left(\left|p^-\right\rangle\!\left\langle k^-\right| + \left|k^+\right\rangle\!\left\langle p^+\right|\right) \tag{H.519}$$

$$\epsilon^-_\mu(p,k)\gamma^\mu = -\frac{\sqrt{2}}{[kp]}\left(\left|k^-\right\rangle\!\left\langle p^-\right| + \left|p^+\right\rangle\!\left\langle k^+\right|\right). \tag{H.520}$$

The products with m in the numerator involve $\not\epsilon^-(3,4)\not\epsilon^+(4,3) = 0$. The products with $\not p_3$ involve

$\not\epsilon^-(3,4)\not p_3\not\epsilon^+(4,3)=0$. We are left with

$$\begin{aligned} i\,A_4(1_q^I,2_{\bar q}^J,3_g^-,4_g^+) &= \left(\frac{i}{\sqrt{2}}\right)^2\frac{2i}{\langle 34\rangle[43](s_{13}-m^2)}\left(\langle 1^{I,-}\big| + \langle 1^{I,+}\big|\right)\\ &\quad\times\left(\big|4^-\rangle\langle 3^-\big| + \big|3^+\rangle\langle 4^+\big|\right)\not p_1\left(\big|4^-\rangle\langle 3^-\big| + \big|3^+\rangle\langle 4^+\big|\right)\left(\big|2^{J,-}\rangle + \big|2^{J,+}\rangle\right)\\ &= \frac{-i}{s_{34}(s_{13}-m^2)}\left(\langle 1^{I,-}\big| + \langle 1^{I,+}\big|\right)\\ &\quad\times\left(\big|4^-\rangle\,\langle 31^K\rangle[1_K4]\,\langle 3^-\big| + \big|3^+\rangle\,[41^K]\langle 1_K3\rangle\,\langle 4^+\big|\right)\left(\big|2^{J,-}\rangle + \big|2^{J,+}\rangle\right)\\ &= \frac{-i\langle 31^K\rangle[1_K4]}{s_{34}(s_{13}-m^2)}\left(-\langle 31^I\rangle[42^J] - [41^I]\langle 32^J\rangle\right) \end{aligned} \tag{H.521}$$

Now we turn to the limit $m\to 0$. We want to see how massive amplitudes for particles with spin decompose into the different helicity components in the massless limit. To do so, it is convenient to expand λ_a^I in a basis of two-dimensional vectors $\zeta^+=(1,0)^T$, $\zeta^-=(0,1)^T$ in the little-group space, as we did in the lecture,

$$\lambda_a^I(p) = \lambda_a(p)(\zeta^+)^I + \eta_a(p)(\zeta^-)^I\,, \tag{H.522}$$

$$\tilde\lambda_{\dot a}^I(p) = \tilde\lambda_{\dot a}(p)(\zeta^-)^I - \tilde\eta_{\dot a}(p)(\zeta^+)^I\,, \tag{H.523}$$

Hence we find

$$iA_4(1_q^I,2_{\bar q}^J,3_g^-,4_g^+) = \frac{i\lambda^a(3)\lambda_a^K(1)\tilde\lambda_{K,\dot b}(1)\tilde\lambda^{\dot b}(4)}{s_{34}(s_{13}-m^2)}\left(\tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\lambda_{\dot e}^J(2)\lambda(3)^d\lambda_d^I(1) + \tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\lambda_{\dot e}^I(1)\lambda(3)^d\lambda_d^J(2)\right) \tag{H.524}$$

first note that

$$\lambda_a^K(1)\tilde\lambda_{K,\dot b}(1) = \lambda_a(1)\tilde\lambda_{\dot b}(1) + \eta_a(1)\tilde\eta_{\dot b}(1) \xrightarrow[m\to 0]{} \lambda_a(1)\tilde\lambda_{\dot b}(1) \tag{H.525}$$

then you look at all components of A^{IJ},

- $I=0, J=0$

$$\left(-\tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\eta_{\dot e}(2)\lambda(3)^d\lambda_d(1) - \tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\eta_{\dot e}(1)\lambda(3)^d\lambda_d(2)\right) \xrightarrow[m\to 0]{} 0 \tag{H.526}$$

- $I=0, J=1$

$$\left(+\tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\lambda_{\dot e}(2)\lambda(3)^d\lambda_d(1) - \tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\eta_{\dot e}(1)\lambda(3)^d\eta_d(2)\right) \xrightarrow[m\to 0]{} \tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\lambda_{\dot e}(2)\lambda(3)^d\lambda_d(1) \tag{H.527}$$

- $I=1, J=0$

$$\left(-\tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\eta_{\dot e}(2)\lambda(3)^d\eta_d(1) + \tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\lambda_{\dot e}(1)\lambda(3)^d\lambda_d(2)\right) \xrightarrow[m\to 0]{} \tilde\lambda_{\dot c}(4)\epsilon^{\dot c\dot e}\tilde\lambda_{\dot e}(1)\lambda(3)^d\lambda_d(2) \tag{H.528}$$

- $I=1, J=1$

$$\left(\tilde{\lambda}_{\dot{c}}(4)\epsilon^{\dot{c}\dot{e}}\tilde{\lambda}_{\dot{e}}(2)\lambda(3)^{d}\eta_{d}(1)+\tilde{\lambda}_{\dot{c}}(4)\epsilon^{\dot{c}\dot{e}}\tilde{\lambda}_{\dot{e}}(1)\lambda(3)^{d}\eta_{d}(2)\right)\xrightarrow[m\to 0]{}0 \tag{H.529}$$

where we used that λ_i scales with $\sqrt{E_i+|\vec{p}_i|}$ (and, by the way, is denoted in the same way as the corresponding spinor for massless particles because they are equal in the high energy limit) and η_i scales with $\sqrt{E_i-|\vec{p}_i|}$ and therefore vanishes in the high energy limit. Finally one should find

$$M\left(1_i^I,2^{J,j},3^-,4^+\right)\xrightarrow[m\to 0]{}ig^2\left[\frac{(T^{a_3}T^{a_4})_j^i}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}+\frac{(T^{a_4}T^{a_3})_j^i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right]\begin{pmatrix}0 & \langle 13\rangle\langle 23\rangle^3\\ -\langle 13\rangle^3\langle 23\rangle & 0\end{pmatrix}. \tag{H.530}$$

H.42 $q\bar{q}\to(n{-}2)$ gluons with massive quarks and all positive-helicity gluons

Prove by induction the following closed-form formula [61] for the n-point colour-stripped amplitude for $q\bar{q}\to(n-2)$ gluons with massive quarks and all positive-helicity gluons,

$$iA_n\left(1_q^I,3_g^+,4_g^+,\ldots,n_g^+,2_{\bar{q}}^J\right)=\frac{im\left\langle 1^I2^J\right\rangle\left[3\left|\prod_{j=3}^{n-2}\left\{\not{P}_{1,j}\not{p}_{j+1}+(s_{13\ldots j}-m^2)\right\}\right|n\right]}{(s_{13}-m^2)\,(s_{134}-m^2)\ldots\left(s_{13\ldots(n-1)}-m^2\right)\langle 34\rangle\langle 45\rangle\ldots\langle n-1\mid n\rangle}. \tag{H.531}$$

1. For the first induction step, derive the expression for $n=4$.

   Solution. We can redo the same steps of the previous exercise and find

$$iA_4(1^I,2^J,3^+,4^+)=\frac{im\langle 1^I2^J\rangle[34]}{(s_{13}-m^2)\langle 34\rangle}. \tag{H.532}$$

2. In order to prove $n-1 \implies n$, use on shell recursion relations with the following shift on the gluonic momenta,

$$\left|\widehat{n-1}\right]=|n-1]-z\,|n]\,, \tag{H.533}$$

$$|\hat{n}\rangle=|n\rangle+z\,|n-1\rangle \tag{H.534}$$

   assuming that with this shift $A_n(z)\to 0$ as $z\to\infty$.

   Hint *The 3-point colour-ordered amplitude with $\epsilon(3,q)$ as polarization vector of the gluon* *takes the form,*

$$iM_3(1^{I,i}, 2^J_j, 3^+) = -\frac{ig(T^{a_3})^i_j}{\sqrt{2}m\langle 3q\rangle}\langle 1^I 2^J\rangle\, [3|\, \not{p}_1\, |q\rangle \tag{H.535}$$

$$iM_3(1^{I,i}, 2^J_j, 3^-) = \frac{ig(T^{a_3})^i_j}{\sqrt{2}m[3q]}[1^I 2^J]\, \langle 3|\, \not{p}_1\, |q] \tag{H.536}$$

Write $M_n \sim \left(\frac{ig}{\sqrt{2}}\right)^{n-2}(T\cdots T)A_n$ *and work out the recursion relation only for the colour-stripped* A_n*, starting from* $A_4(1^I_q, 2^J_{\bar{q}}, 3^+_g, 4^+_g)$*.*

Solution. Here we are dealing with colour-stripped partial amplitudes, the order of the labels of the gluons in A_n is the same as their order in the corresponding subamplitudes on the r.h.s. of the recursion relation. Moreover, the contributions to the r.h.s. can come from residues of simple poles resulting from an internal propagator going on shell. Here, the internal propagator could be either a gluon propagator or a massive quark propagator. We claim that, in the latter case, the BCFW relations look like

$$A_n(p_1,\cdots,p_n) = \sum_k \sum_{\substack{h=\text{states}\\ \text{propagating}\\ \text{particle}}} A_{k+1}(p_1,\cdots,\hat{p}_i,\cdots,-\widehat{P}^h_{1,k})\frac{1}{P^2_{1,k}-m^2}A_{n-k+1}(\widehat{P}^{-h}_{1,k}, p_{k+1},\cdots,\hat{p}_i,\cdots,p_n). \tag{H.537}$$

The sum of the residues on the r.h.s. of the recursion relation is over partitions of the external particles, but also over all internal states (helicity, mass, etc.), because one needs to sum over all possible on-shell states of the intermediate particle. It might be confusing at first sight the fact that the standard recursion relation has the intermediate propagator of the form $1/(p^2-m^2)$ because the fermion propagator is like $1/p$ not $1/p^2$. We recover the correct scaling of the internal fermion propagator when we sum over all of the spin states of the internal propagating particle,

$$\begin{aligned}&\sum_s A_{k+1}(p_1,\cdots,\hat{p}_i,\cdots,-\widehat{P}^s_{(q);1,k})\frac{1}{P^2_{1,k}-m^2}A_{n-k+1}(\widehat{P}^{-s}_{(\bar{q});1,k}, p_{k+1},\cdots,\hat{p}_i,\cdots,p_n)\\ &= A_{k+1}(p_1,\cdots,\hat{p}_i,\cdots,-\widehat{P}^*_{1,k})\frac{\not{P}_{1,k}+m}{P^2_{1,k}-m^2}A_{n-k+1}(\widehat{P}^*_{1,k}, p_{k+1},\cdots,\hat{p}_i,\cdots,p_n)\end{aligned} \tag{H.538}$$

where P^* indicates the external spinor wave-function has been stripped off this amplitude, and we used the conventional spin sums, (up to coefficients in front coming from analytic continuation of the spinor with the negative momentum):

$$\sum_{s=1,2} u_s(p)\bar{u}_s(p) = \not{p} + m\,, \tag{H.539}$$

$$\sum_{s=1,2} v_s(p)\bar{v}_s(p) = \not{p} - m\,. \tag{H.540}$$

Next, each diagram is associated with a particular value for the complex parameter z, which can again be found via the condition that the internal propagator is on shell, i.e. for massive particles we need the solution $z = z_k$ to the equation $\hat{P}^2(z) - m^2 = 0$. Since $\hat{P}(z)^\mu = P^\mu - zq^\mu$, the pole is at

$$z_k = \frac{P^2 - m^2}{2P \cdot q}.$$

If we use the spinor-helicity formalism with little group indices, the sum over all spin states of H.538 is performed as a sum over all possible values of the little group indices. Of course, the original amplitude did not carry the extra little group index of the massive intermediate particle produced in the factorization sum and that is why the extra little group indices are contracted in such a way to create a $SU(2)$ invariant (i.e. if you have $-\hat{P}^K$ in one subamplitude, then $\hat{P}_K$ is in the other). Finally, $\hat{P}(z)$ only depends on z if the shifted momenta belong to opposite sides from the internal propagator going on shell, so the only contributions to the on-shell relation with this shift are:

$$\begin{aligned} iA_n(1^I, 3^+, 4^+, \cdots, n^+, 2^J) &= iA_{n-1}(1^I, 3^+, 4^+, \cdots, (\widehat{n-1})^+, -\hat{p}^K_{n2}) \frac{i}{s_{n2} - m^2} iA_3(\hat{p}_{n2;K}, \hat{n}^+, 2^J) \\ &+ \sum_h iA_3((n-2)^+, (\widehat{n-1})^+, -\hat{P}^{-h}_{n-1,n-2}) \frac{i}{s_{(n-2)(n-1)}} iA_{n-1}(1^I, 3^+, 4^+, \cdots, (n-3)^+, \hat{P}^{+h}_{n-1,n-2}, \hat{n}, 2^J)\,, \end{aligned} \tag{H.541}$$

(with $p_{ij} := p_i + p_j$). We start by considering the second term. Firstly, we note that only the contribution $h = +$ remains. Next, the value of z at which we evaluate the residue is

$$z = \frac{[n-2|n-1]}{[n-2|n]} \tag{H.542}$$

so

$$[n-2|\widehat{n-1}] = [n-2|n-1] - \frac{[n-2|n-1]}{[n-2|n]}[n-2|n] = 0 \tag{H.543}$$

and therefore $iA_3((n-2)^+, (\widehat{n-1})^+, -\hat{P}^-_{n-1,n-2}) = 0$. For the remaining contribution in eq. H.541 we can use the induction hypothesis for $n-1$, and we get

$$A_n\left(1^I, 3^+, \ldots, n^+, 2^J\right) = \frac{i\left\langle 1^I \mid -\hat{p}^K_{n2}\right\rangle \left[3\left|\prod_{j=3}^{n-3}\left\{\not{P}_{1,j}\not{p}_{j+1} + (s_{13\ldots j} - m^2)\right\}\right|\widehat{n-1}\right] \left\langle \hat{p}_{n2;K} 2^J\right\rangle [n|\hat{\not{p}}_{n2}|q\rangle}{(s_{13} - m^2)\ldots\left(s_{13\ldots.(n-2)} - m^2\right)(s_{2n} - m^2)\langle 34\rangle \ldots \langle n-2 \mid n-1\rangle \langle \hat{n} q\rangle} \tag{H.544}$$

where q is the reference of the polarization vector of the gluon in the 3-point amplitude eq. H.535, and we are free to set $q = p_{n-1}$. In the numerator you have

$$[n|\hat{\not{p}}_{n2}|n-1\rangle = [n|\left(\not{p}_{n2} + z\not{q}\right)|n-1\rangle = \langle n-1|\not{p}_2|n],$$

where this time q^μ is the momentum of the shift and we used that either $\not{q} = \lambda_{n-1}\tilde{\lambda}_n$ or

$\not{q} = \lambda_n \tilde{\lambda}_{n-1}$, and here $[n|\not{q}|n-1\rangle = 0$, together with $[n|\not{p}_n|n-1\rangle = 0$. So we write

$$iA_n\left(1^I, 3^+, \ldots, n^+, 2^J\right) = \frac{im\left\langle 1^I 2^J\right\rangle \left[3\left|\prod_{j=3}^{n-3}\left\{\not{P}_{1,j}\not{p}_{j+1} + (s_{13\ldots j} - m^2)\right\}\right|\widehat{n-1}\right]\langle n-1|\not{p}_2|n]}{(s_{13}-m^2)\ldots\left(s_{13\ldots(n-1)}-m^2\right)\langle 34\rangle\ldots\langle n-2\mid n-1\rangle\langle n-1\mid n\rangle}, \tag{H.545}$$

where we used that

$$\left\langle 1^I \mid -\hat{p}^K_{n2}\right\rangle\left\langle \hat{p}_{n2;K}2^J\right\rangle = m\left\langle 1^I 2^J\right\rangle, \tag{H.546}$$

this can be seen using the Schouten identity (1.53), $\langle pk\rangle\langle qv\rangle + \langle pq\rangle\langle vk\rangle + \langle pv\rangle\langle kq\rangle = 0$, together with antisymmetry of the SU(2) contraction, in the following way. First, suppose

$$\mid -\hat{p}^K_{n2}\rangle = a\mid \hat{p}^K_{n2}\rangle$$

with $a \in \mathbb{C}$, with $|a|^2 = 1$, some coefficient resulting from analytic continuation as usual. Then

$$\left\langle 1^I \mid -\hat{p}^K_{n2}\right\rangle\left\langle \hat{p}_{n2;K}2^J\right\rangle = a\left\langle 1^I \hat{p}^K_{n2}\right\rangle\left\langle \hat{p}_{n2;K}2^J\right\rangle = a\left(-\left\langle 1^I \hat{p}_{n2K}\right\rangle\left\langle 2^J \hat{p}^K_{n2}\right\rangle - \left\langle 1^I 2^J\right\rangle\left\langle \hat{p}^K_{n2}\hat{p}_{n2K}\right\rangle\right) \tag{H.547}$$

so:

$$\begin{aligned} a\left\langle 1^I \hat{p}^K_{n2}\right\rangle\left\langle \hat{p}_{n2;K}2^J\right\rangle - a\left\langle 1^I \hat{p}_{n2K}\right\rangle\left\langle \hat{p}^K_{n2}2^J\right\rangle &= -a\left(-\left\langle 1^I \hat{p}^M_{n2}\right\rangle\left\langle \hat{p}^N_{n2}2^J\right\rangle + \left\langle 1^I \hat{p}^N_{n2}\right\rangle\left\langle \hat{p}^M_{n2}2^J\right\rangle\right)\epsilon_{NM} \\ &= -2a\left\langle 1^I \hat{p}^N_{n2}\right\rangle\left\langle \hat{p}^M_{n2}2^J\right\rangle\epsilon_{NM} = 2a\left\langle 1^I \hat{p}^K_{n2}\right\rangle\left\langle \hat{p}_{n2;K}2^J\right\rangle \\ &= 2\left\langle 1^I \mid -\hat{p}^K_{n2}\right\rangle\left\langle \hat{p}_{n2;K}2^J\right\rangle \\ &= -a\left\langle 1^I 2^J\right\rangle\left\langle \hat{p}^K_{n2}\hat{p}_{n2K}\right\rangle = -a\left\langle 1^I 2^J\right\rangle m\delta^K_K \\ &= -2am\left\langle 1^I 2^J\right\rangle. \end{aligned} \tag{H.548}$$

If we choose $a = -1$, we will obtain the correct sign in eq. (H.545). Next, the r.h.s. is evaluated at

$$z = -\frac{(p_2+p_n)^2 - m^2}{\langle n-1|\not{p}_2 + \not{p}_n|n]} = -\frac{p_2^2 - m^2 + 2p_2\cdot p_n + p_n^2}{\langle n-1|\not{p}_2|n]} = -\frac{2\langle n2^M\rangle[2_M n]}{\langle n-1|\not{p}_2|n]} = -\frac{\langle n|\not{p}_2|n]}{\langle n-1|\not{p}_2|n]} \tag{H.549}$$

and in the numerator you find

$$
\begin{aligned}
\left|\widehat{n-1}\right] \langle n-1|\not{p}_2|n] &= \left(|n-1] + \frac{\langle n|\not{p}_2|n]}{\langle n-1|\not{p}_2|n]}|n]\right)\langle n-1|\not{p}_2|n] \\
&= |n-1]\langle n-1|\not{p}_2|n] + |n]\langle n|\not{p}_2|n] \\
&= \not{p}_{n(n-1)}\not{p}_2|n] \\
&= \left(\not{p}_{(n-1)} + (-\not{p}_{n-1} - \not{p}_2 - \not{p}_{13\cdots(n-2)})\right)\not{p}_2|n] \\
&= \left(-\not{p}_2\not{p}_2 - \not{p}_{13\cdots(n-2)}\not{p}_2\right)|n] \\
&= \left(-m^2 - \not{p}_{13\cdots(n-2)}\left(-\not{p}_{13\cdots(n-2)} - \not{p}_{n-1} - \not{p}_n\right)\right)|n] \\
&= \left(\not{p}_{13\cdots(n-2)}\not{p}_{n-1} + (s_{13\cdots(n-2)} - m^2)\right)|n]\,,
\end{aligned}
\tag{H.550}
$$

where we used repeatedly momentum conservation, $\not{p}_n|n] = 0$ and $\not{p}\not{p} = p^2\mathbb{1}$. This completes the proof.

H.43 Linear relations among scattering equations

For $f(z_i,p) = \sum_{j\neq i}\frac{2p_i\cdot p_j}{z_i - z_j}$ prove that $\sum_{i=1}^{n} z_i^m f_i(z,p) = 0$ for $m = 0,1,2$.

Solution. We have

$$
m=0: \qquad \sum_{i=1}^{n} f(z_i,p) = \sum_i\sum_{j\neq i}\frac{2p_i\cdot p_j}{z_i - z_j} = 0\,, \tag{H.551}
$$

because of the antisymmetry of the denominators.

The case $m=1$ is

$$
\begin{aligned}
m=1: \qquad \sum_{i=1}^{n} z_i f(z_i,p) &= \sum_{i=1}^{n}\sum_{j\neq i}((z_i - z_j) + z_j)\frac{2p_i\cdot p_j}{z_i - z_j} \\
&= \underbrace{\sum_i\sum_{j\neq i} 2p_i p_j}_{p_{i\mu}\sum_{j\neq i}p_j^{\mu} = -p_i^2 = 0} + \sum_i\sum_{j\neq i} z_j \underbrace{\frac{2p_i\cdot p_j}{z_i - z_j}}_{=:f_{(i,j)}} \\
(\text{relabel } i\leftrightarrow j) &= \sum_i\sum_{j\neq i} z_i f_{(j,i)} \\
(f_{(j,i)} = -f_{(i,j)}) &= -\sum_i\sum_{j\neq i} z_i f_{(i,j)} \\
&= -\sum_{i=1}^{n} z_i f(z_i,p)\,.
\end{aligned}
\tag{H.552}
$$

The case $m = 2$ is

$$m=2: \qquad \sum_{i=1}^n z_i^2 f(z_i,p) = \sum_{i=1}^n \sum_{j\neq i} (z_i)^2 \frac{2p_i\cdot p_j}{z_i - z_j}$$

$$= \sum_{i=1}\sum_{j\neq i}((z_i)^2 - (z_j)^2)\frac{2p_i\cdot p_j}{z_i-z_j} + \overbrace{\sum_{i=1}\sum_{j\neq i}(z_j)^2\frac{2p_i\cdot p_j}{z_i-z_j}}^{\text{as above:}-\sum_{i=1}^n z_i^2 f(z_i,p)}$$

$$= \sum_{i=1}\sum_{j\neq i}(z_i+z_j)(2p_i\cdot p_j) - \sum_{i=1}^n z_i^2 f(z_i,p)$$

$$= 2\sum_{i=1} z_i \underbrace{\sum_{j\neq i}(2p_i\cdot p_j)}_{=-2p_i^2=0} - \sum_{i=1}^n z_i^2 f(z_i,p)$$

$$= -\sum_{i=1}^n z_i^2 f(z_i,p)\,. \tag{H.553}$$

H.44 $PSL(2,\mathbb{C})$ invariance of scattering equations

Show that if $(z_1,\ldots,z_n)$ is a solution of the scattering equations, also $(z_1',\ldots,z_n')$ with

$$z' = \frac{az+b}{cz+d}, \qquad \{a,b,c,d\in\mathbb{C}\,|\,ad-bc=1\}\,, \tag{H.554}$$

is a solution.

Solution. We have:

$$z_i' - z_j' = \frac{(az_i+b)(cz_j+d)-(az_j+b)(cz_i+d)}{(cz_i+d)(cz_j+d)} = \frac{\overbrace{(ad-bc)}^{=1}(z_i-z_j)}{(cz_i+d)(cz_j+d)}\,, \tag{H.555}$$

and can write the inverse as

$$\frac{1}{z_i'-z_j'} = \frac{(cz_i+d)}{z_i-z_j}(cz_i+d-c(z_i-z_j)) = \frac{(cz_i+d)^2}{z_i-z_j} - c(cz_i+d)\,. \tag{H.556}$$

We set $x=(cz_i+d)^2$ and $y=-c(cz_i+d)$ where x and y do not depend on z_j. Then

$$\sum_{j\neq i}\frac{2p_i\cdot p_j}{z_i'-z_j'} = \sum_{j\neq i}\frac{2p_i\cdot p_j}{z_i-z_j}x + \sum_{j\neq i}2p_i\cdot p_j y\,, \tag{H.557}$$

where the first term vanishes because $(z_1,\ldots,z_n)$ is a solution of the scattering equation and the second term vanishes because of momentum conservation and on-shellness,

$$2y\; p_{i\mu}\sum_{j=1,j\neq i}^n p_j^\mu = -2yp_i^2 = 0\,, \tag{H.558}$$

which proves the statement.

H.45 The $PSL(2,\mathbb{C})$-invariant function $U(z,p)$

Let us consider the $PSL(2,\mathbb{C})$-invariant function,

$$U(z,p) = \prod_{j<k} (z_j - z_k)^{2p_j p_k} \,. \tag{H.559}$$

Show that $U^{-1}\partial_{z_i} U = f_i(z,p)$

Solution. First notice that $\prod_{j<k}$ can be written as $\prod_{j=1}^{n-1}\prod_{k=j+1}^{n}$ or $\prod_{k=1}^{n-1}\prod_{j=1}^{k-1}$. So we have

$$\begin{aligned}
\partial_{z_i} U &= \partial_{z_i} \prod_{j<k} (z_j - z_k)^{2(p_i p_j)} \\
&= \prod_{k=i+1}^{n} \frac{2(p_i p_j)}{z_i - z_k} U - \prod_{j=1}^{i-1} \frac{2(p_i p_j)}{z_j - z_i} U \\
&= \sum_{j\neq i} \frac{2(p_i p_j)}{z_i - z_j} U \\
&= f_i(z,p) U \,.
\end{aligned} \tag{H.560}$$

H.46 Infinitesimal Möbius transformations

Parametrize the Möbius transformations $z \mapsto g(z) = \dfrac{az+b}{cz+d}$ through a parameter $t \in [0,1]$ such that for $t=0$ $a=d=1$ and $b=c=0$ and $g_0(z)=z$. Compute an infinitesimal Möbius transformation $\delta g(z) = \delta z$ considering an infinitesimal shift on the entries of the Möbius transformation.

Solution.

$$\begin{aligned}
\delta g(z) &= \left.\frac{(\delta a z + \delta b)(cz+d) - (az+b)(\delta c z + \delta b)}{(cz+d)^2}\right|_{t=0} \\
&= \delta a z + \delta b - z(\delta c z + \delta d) \\
&= -\delta c z^2 + (\delta a - \delta d) z + \delta b = \delta z \,.
\end{aligned} \tag{H.561}$$

H.47 The Pfaffian and the three- and four-gluon amplitudes

Consider the $2n \times 2n$ antisymmetric matrix,

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \tag{H.562}$$

with

$$A_{ab} = \begin{cases} \dfrac{2(p_a p_b)}{z_{ab}} & a \neq b \\ 0 & a = b \end{cases} \qquad B_{ab} = \begin{cases} \dfrac{2(\epsilon_a \epsilon_b)}{z_{ab}} & a \neq b \\ 0 & a = b \end{cases} \qquad C_{ab} = \begin{cases} \dfrac{2(\epsilon_a p_b)}{z_{ab}} & a \neq b \\ -\sum\limits_{j=1, j\neq a}^{n} \dfrac{2(\epsilon_a p_j)}{z_{aj}} & a = b \end{cases} \tag{H.563}$$

where $\epsilon_a = \epsilon(p_a)$ are the polarization vectors and $z_{ab} = z_a - z_b$. Show that the Pfaffian of Ψ vanishes.

i) Show that the Pfaffian of Ψ vanishes on the scattering equations.

Solution. We see that the sum of all rows of the matrix $-C^T$ is identically 0, whereas the sum of all rows of A gives the scattering equations. That means, on the solutions of the scattering equations the rows of the matrix $(A, -C^T)$ are linearly dependent and the determinant and therefore the Pfaffian of Ψ vanishes.

Consider the three-gluon amplitude, $A(1^{\lambda_1}, 1^{\lambda_2}, 1^{\lambda_3})$.

ii) Compute the polarization factor,

$$E(p, \epsilon, z) = \frac{(-1)^{i+j}}{2^{n/2} z_{ij}} \mathrm{Pf}\Psi^{ij}_{ij}\,, \tag{H.564}$$

and verify, that it has at most logarithmic singularities in z_{ij}.

Solution. *For this and the following exercises for the four-gluon amplitude, you can use the Mathematica notebook enclosed in the Tutorials to verify the computation and try out different expressions.*

We have

$$\begin{aligned} E(p, \epsilon, z) &= \frac{(-1)^3}{2^{3/2} z_{12}} \mathrm{Pf}\Psi^{12}_{12} \\ &= \frac{\sqrt{2}\,(\epsilon_1 \cdot \epsilon_3 p_3 \cdot \epsilon_2 - \epsilon_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1)}{z_{1,2} z_{1,3} z_{2,3}} - \frac{\sqrt{2}\epsilon_1 \cdot \epsilon_2\,(z_{1,3} p_2 \cdot \epsilon_3 + z_{2,3} p_1 \cdot \epsilon_3)}{z_{1,2}^2 z_{1,3} z_{2,3}} \\ &= \frac{\sqrt{2}\,(\epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 + \epsilon_1 \cdot \epsilon_3 p_3 \cdot \epsilon_2 - \epsilon_2 \cdot \epsilon_3 p_3 \cdot \epsilon_1)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} \\ &= \frac{\sqrt{2}\,(\epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 + \epsilon_1 \cdot \epsilon_3 p_3 \cdot \epsilon_2 + \epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_1)}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}. \end{aligned} \tag{H.565}$$

To go from the second to the third line we used momentum conservation $\epsilon_3 \cdot p_2 = -\epsilon_3 \cdot p_1$. Only once we impose it, we see the singularity structure is manifestly logarithmic. If we furthermore use $-p_3 \cdot \epsilon_1 = p_2 \cdot \epsilon_1$, we can make the solution manifestly cyclic invariant (this is always possible, if no reference vector is specified).

iii) Compute the three-gluon form factor and verify that it indeed is independent on the z_i.

Solution. We have

$$J(z,p) = z_{1,2}^2 z_{1,3}^2 z_{2,3}^2\,, \tag{H.566}$$

$$C(\sigma,z) = -\frac{1}{z_{1,2}z_{1,3}z_{2,3}}\,, \tag{H.567}$$

and see that the complete z-dependence will cancel and we are left with the three-gluon vertex.

We now want to compute the MHV-amplitude $A(1^-,2^-,3^+,4^+)$ with the CHY-formalism.

iv) Compute the polarization factor and verify that it is has at most single poles.

Hint Depending on the reduced Pfaffian you choose for the polarization factor, a specific choice of the reference vectors of your polarizations will make the expression tangible.

Solution. We choose to compute the polarization factor[4],

$$E(p,\epsilon,z) = \frac{(-1)^3}{4z_{12}}\mathrm{Pf}\Psi_{12}^{12}\,. \tag{H.568}$$

Since we want to remove the first and second row and column, we will have to deal with the non-trivial C_{33} and C_{44}. To make them simpler, we choose the polarizations $\epsilon^-(1,2)$, $\epsilon^-(2,1)$, $\epsilon_3^+(3,1)$ and $\epsilon_4^+(4,1)$. This will make the products $\epsilon^+(i,1)p_1 = \epsilon^+(i,1)\epsilon^-(1,j) = 0$ vanish and simplifies the computation. We find

$$\begin{aligned}
E(p,\epsilon,z) =& \frac{(-1)^3}{4z_{12}}\mathrm{Pf}\Psi_{12}^{12} \\
=& -\frac{2\epsilon_{1,2}^- \cdot \epsilon_{2,1}^- p_2 \cdot \epsilon_{3,1}^+ p_2 \cdot \epsilon_{4,1}^+}{z_{1,2}^2 z_{2,3} z_{2,4}} - \frac{2\epsilon_{1,2} \cdot \epsilon_{2,1}^- p_2 \cdot \epsilon_{3,1}^+ p_3 \cdot \epsilon_{4,1}^+}{z_{1,2}^2 z_{2,3} z_{3,4}} + \frac{2\epsilon_{1,2}^- \cdot \epsilon_{2,1}^- p_2 \cdot \epsilon_{4,1}^+ p_4 \cdot \epsilon_{3,1}^+}{z_{1,2}^2 z_{2,4} z_{3,4}} \\
&- \frac{2\epsilon_{2,1}^- \cdot \epsilon_{3,1}^+ p_2 \cdot \epsilon_{4,1}^+ p_3 \cdot \epsilon_{1,2}^-}{z_{1,2} z_{1,3} z_{2,3} z_{2,4}} - \frac{2\epsilon_{2,1}^- \cdot \epsilon_{4,1}^+ p_2 \cdot \epsilon_{3,1}^+ p_4 \cdot \epsilon_{1,2}}{z_{1,2} z_{1,4} z_{2,3} z_{2,4}} - \frac{2\epsilon_{2,1}^- \cdot \epsilon_{3,1}^+ p_3 \cdot \epsilon_{1,2}^- p_3 \cdot \epsilon_{4,1}^+}{z_{1,2} z_{1,3} z_{2,3} z_{3,4}} \\
&- \frac{2\epsilon_{2,1}^- \cdot \epsilon_{3,1}^+ p_3 \cdot \epsilon_{4,1}^+ p_4 \cdot \epsilon_{1,2}^-}{z_{1,2} z_{1,4} z_{2,3} z_{3,4}} + \frac{2\epsilon_{2,1}^- \cdot \epsilon_{4,1}^+ p_3 \cdot \epsilon_{1,2}^- p_4 \cdot \epsilon_{3,1}^+}{z_{1,2} z_{1,3} z_{2,4} z_{3,4}} + \frac{2\epsilon_{2,1}^- \cdot \epsilon_{4,1}^+ p_4 \cdot \epsilon_{1,2}^- p_4 \cdot \epsilon_{3,1}^+}{z_{1,2} z_{1,4} z_{2,4} z_{3,4}} \\
=& \frac{2p_4 \cdot \epsilon_{1,2}^- \left(\epsilon_{2,1}^- \cdot \epsilon_{3,1}^+ p_2 \cdot \epsilon_{4,1}^+ - \epsilon_{2,1}^- \cdot \epsilon_{4,1}^+ p_2 \cdot \epsilon_{3,1}\right)}{(z_1-z_2)(z_1-z_3)(z_2-z_3)(z_4-z_1)} + \frac{2p_3 \cdot \epsilon_{1,2}^- \left(\epsilon_{2,1}^- \cdot \epsilon_{3,1}^+ p_2 \cdot \epsilon_{4,1}^+ - \epsilon_{2,1}^- \cdot \epsilon_{4,1}^+ p_2 \cdot \epsilon_{3,1}\right)}{(z_1-z_2)(z_1-z_3)(z_2-z_3)(z_4-z_2)} \\
&- \frac{2\left(p_3 \cdot \epsilon_{1,2} + p_4 \cdot \epsilon_{1,2}^-\right)\left(\epsilon_{2,1}^- \cdot \epsilon_{3,1}^+ p_2 \cdot \epsilon_{4,1}^+ - \epsilon_{2,1}^- \cdot \epsilon_{4,1}^+ p_2 \cdot \epsilon_{3,1}^+\right)}{(z_1-z_2)(z_1-z_3)(z_2-z_3)(z_4-z_3)}\,,
\end{aligned} \tag{H.569}$$

where again only after applying momentum conservation and z-partial-fractioning the double pole vanishes.

v) Compute the Jacobian and the Parke-Taylor factor.

Hint Φ_{234}^{134} will give a nice form.

[4] It can be found without specified polarizations in https://arxiv.org/pdf/1610.05318.pdf eq. (585). However, its not so trivial to obtain it, since momentum conservation and z-partial-fractioning has to be applied to a relatively large expression. You can try it out on the notebook enclosed in the Tutorials.

Solution. Taking the suggestion from the hint we have

$$J(z,p) = -\frac{z_{1,2}^2 z_{1,3} z_{1,4} z_{2,3} z_{2,4} z_{3,4}^2}{2p_1 \cdot p_2}\,. \tag{H.570}$$

Choosing e.g. Φ^{123}_{123} for the computation, one gets

$$J(z,p) = \frac{z_{1,2}^2 z_{1,3}^2 z_{2,3}^2}{-\dfrac{2p_1 \cdot p_4}{z_{1,4}^2} - \dfrac{2p_2 \cdot p_4}{z_{2,4}^2} - \dfrac{2p_3 \cdot p_4}{z_{3,4}^2}}\,, \tag{H.571}$$

which is already not so nice looking. The Parke-Taylor factor is

$$C(1234, z) = -\frac{1}{z_{1,2} z_{1,4} z_{2,3} z_{3,4}}\,. \tag{H.572}$$

vi) Solve the scattering equation and compute $A(1^-, 2^-, 3^+, 4^+)$ in terms of spinor products. Verify also, that $J(z,p)C(1234,z)E(z,p,\epsilon,z)$ depends only on cross-ratios.

Solution. We take $z_2 = 0$, $z_3 = 1$ and $z_4 = \infty$, which determines $z_1 = -\dfrac{s}{u}$ where $u = 2p_1 \cdot p_4 = 2p_2 \cdot p_3$. The complete amplitude reads

$$\begin{aligned} A(1^-, 2^-, 3^+, 4^+) &= -\frac{(z_1 - z_2)(z_3 - z_4)\, p_3 \cdot \epsilon^-_{1,2} \left(\epsilon^-_{2,1} \cdot \epsilon^+_{3,1} p_2 \cdot \epsilon^+_{4,1} - \epsilon_{2,1} \cdot \epsilon^+_{4,1} p_2 \cdot \epsilon^+_{3,1}\right)}{p_1 \cdot p_2 (z_2 - z_3)(z_1 - z_4)}\Bigg|_{\substack{z_1 = -\frac{s}{u} \\ z_2 = 0 \\ z_3 = 1 \\ z_4 = \infty}} \\ &= -\frac{2p_3 \cdot \epsilon^-_{1,2}}{u} \left(\epsilon^-_{2,1} \cdot \epsilon^+_{3,1} p_2 \cdot \epsilon_{4,1} - \epsilon^-_{2,1} \cdot \epsilon^+_{4,1} p_2 \cdot \epsilon^+_{3,1}\right)\,, \end{aligned} \tag{H.573}$$

and note in particular that it is manifestly $PSL(2,\mathbb{C})$ invariant, since it only depends on a cross-ratio. In order to compute it in terms of spinor brackets, we will use

$$\epsilon^-_{2,1} \cdot \epsilon^+_{j,1} = \frac{\langle 21[j1]\rangle}{[21]\langle 1j\rangle}\,, \tag{H.574}$$

$$[23]\langle 34\rangle = \langle 4|\not{3}|2] = -\langle 41\rangle[12]\,, \tag{H.575}$$

$$[24]\langle 43\rangle = \langle 3|\not{4}|2] = -\langle 31\rangle[12]\,, \tag{H.576}$$

$$s_{13} + s_{14} = t + u = -s\,. \tag{H.577}$$

We have

$$
\begin{aligned}
A(1^-,2^-,3^+,4^+) &= 2\frac{\cancelto{-1}{[23]}\cancelto{-1}{\langle 31\rangle}}{\sqrt{2}[12]\langle 23\rangle\cancel{[32]}}\frac{1}{\sqrt{2}}\Big(\frac{\langle 12\rangle[24]}{\langle 14\rangle}\frac{\langle 21\rangle[31]}{[21]\cancel{\langle 13\rangle}}-\frac{\langle 12\rangle[23]}{\cancel{\langle 13\rangle}}\frac{\langle 21\rangle[41]}{[21]\langle 14\rangle}\Big)\\
&= \frac{\langle 12\rangle^2}{\langle 23\rangle\langle 14\rangle[12]^2}\Big([31][24]-[23][41]\Big)\\
&\overset{\substack{(\text{H}.575)\\(\text{H}.576)}}{=} \frac{\langle 12\rangle^2}{\langle 23\rangle\langle 14\rangle[12]^{\cancel{2}}}\Big(-\frac{\langle 31\rangle\cancelto{-1}{[12]}[31]}{\langle 43\rangle}-\frac{\langle 41\rangle\cancelto{-1}{[12]}[41]}{\langle 43\rangle}\Big)\\
&\overset{(\text{H}.577)}{=} \frac{\langle 12\rangle^2}{\langle 23\rangle\langle 14\rangle\cancel{[12]}\langle 43\rangle}\Big(-\langle 12\rangle\cancelto{-1}{[21]}\Big)\\
&= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,.
\end{aligned}
\tag{H.578}
$$

H.48 One-mass box from a five-point MHV

Study generalised unitarity through an explicit example and construct one of the box-coefficients for the five-gluon one-loop amplitude with helicity configuration $A_5^{1-\text{ loop}}(1^-,2^-,3^+,4^+,5^+)$ [9]. As the number of external legs is $n=5$ the quadruple cut box will factor into one four-point amplitude and three three-point amplitudes. We may then write the decomposition,

$$
A_5^{1-\text{ loop}} = d_{12}I(s_{12})+d_{23}I(s_{23})+d_{34}I(s_{34})+d_{45}I(s_{45})+d_{51}I(s_{51})\,, \tag{H.579}
$$

where the box-integral $I(s_{ij})$ has the massive leg $2p_i\cdot p_j$ arising from two inflowing momenta p_i, p_j attached to the four-point corner. The reflection,

$$
A_5^{1\text{-loop}}(1^-\ 2^-\ 3^+\ 4^+\ 5^+) = -A_5^{1\text{-loop}}(2^-\ 1^-\ 5^+\ 4^+\ 3^+) \tag{H.580}
$$

relates the coefficients d_{51} to d_{23}, and d_{45} to d_{34}, and one needs to compute only three mass-boxes coefficients: d_{12}, d_{23} and d_{34}. In the following we will only consider d_{12}.

1. List the possible helicity configurations for the inner legs after the quadrupole cut applied to the box $I(s_{12})$.

   Solution. By starting from the vertex with 1^-, 2^-, we understand that the two internal legs attached to it must have both helicity $+$, and the internal propagating particles must be glouns. The tree amplitudes for two spin 1/2 fermions (or two scalars) and two identical helicity gluons vanish. Therefore this box coefficient receives contributions only from the gluon loop. At first sight, considering the non-vanishing three-point gluon amplitudes, we can have the helicity configurations of Figs. H.17, H.18, H.19. Let us look in particular at Fig.H.17:

   - at the p_3 corner, we need

   $$\lambda_{\ell_3}\sim\lambda_{\ell_4}\sim\lambda_3,$$

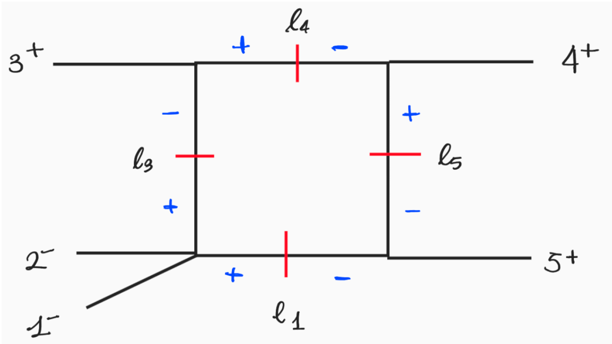


Figure H.17: Possible helicity assignment of internal cut lines on the quadrupole cut in $I(s_{12})$.

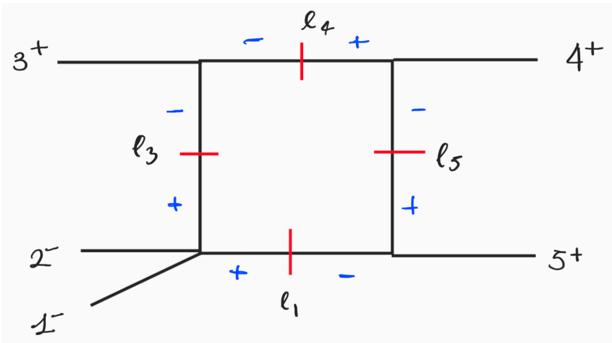


Figure H.18: Possible helicity assignment of internal cut lines on the quadrupole cut in $I(s_{12})$.

- at the p_4 corner, we need

$$\lambda_{\ell_4} \sim \lambda_{\ell_5} \sim \lambda_4.$$

This means that λ_3 needs to be proportional to λ_4, which is not true for generic external momenta (it would be true if p_3 and p_4 were parallel, which only happens in the collinear limit). For $\lambda_3 \not\sim \lambda_4$, the corresponding contribution to the amplitude has to vanish. We can repeat the same reasoning for Fig. H.18, and we find it vanishes as well. In general, we can draw the conclusion that three-gluon box corners must have opposite helicity types (i.e. three-point tree-amplitudes of type MHV can be adjacent to three-point tree-amplitudes of type $\overline{\text{MHV}}$ only) in order to give a nonvanishing product of amplitudes. Hence only Fig. H.19 (where the three-gluon amplitudes are ordered as $(++-), (-+-), (++-)$) contributes to the result.

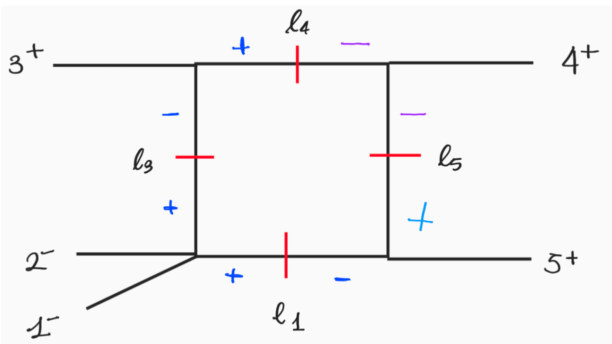


Figure H.19: Possible helicity assignment of internal cut lines on the quadrupole cut in $I(s_{12})$.

2. As seen in the lecture,

$$d_i = \frac{d_i^+ + d_i^-}{2}\,, \tag{H.581}$$

with

$$d_i^\pm = A^{\rm tree}_{(1)}(\ell^\pm)\; A^{\rm tree}_{(2)}(\ell^\pm)\; A^{\rm tree}_{(3)}(\ell^\pm)\; A^{\rm tree}_{(4)}(\ell^\pm)\,, \tag{H.582}$$

where

$$A^{\rm tree}_{(i)}(\ell) := A^{\rm tree}_{n_i+2}(-\ell_i, p_1^{(i)}, \cdots, p_{n_i}^{(i)}, \ell_{i+1}) \tag{H.583}$$

and $\{p_1^{(i)}, \cdots, p_{n_i}^{(i)}\}$ are the external momenta elements of the cluster K_i under consideration, with $\sum_{j=1}^{n_i} p_j^{(i)} = K_i$. Compute d_{12} leaving the cut loop momenta implicit and show that

$$d_{12} = -\frac{1}{2}\frac{\langle 12\rangle^3 \left\langle 3^+ \left|\not\ell_4 \not\ell_5\right| 5^-\right\rangle^3}{\left\langle 2^- \left|\not\ell_3\right| 3^-\right\rangle \left\langle 4^- \left|\not\ell_4 \not\ell_3 \not\ell_1\right| 5^-\right\rangle \left\langle 1^- \left|\not\ell_1 \not\ell_5\right| 4^+\right\rangle} \tag{H.584}$$

Solution.

$$\begin{aligned}
d_{(12)} &= \frac{1}{2} A_4^{\rm tree}\left(-\ell_1^+, 1^-, 2^-, \ell_3^+\right) A_3^{\rm tree}\left(-\ell_3^-, 3^+, \ell_4^+\right) A_3^{\rm tree}\left(-\ell_4^-, 4^+, \ell_5^-\right) A_3^{\rm tree}\left(-\ell_5^+, 5^+, \ell_1^-\right) \\
&= \frac{1}{2}\frac{\langle 12\rangle^3}{\langle 2\ell_3\rangle \langle \ell_3 (-\ell_1)\rangle \langle (-\ell_1) 1\rangle} \frac{[3\ell_4]^3}{[\ell_4 (-\ell_3)]\,[(-\ell_3) 3]} \frac{\langle \ell_5 (-\ell_4)\rangle^3}{\langle 4\ell_5\rangle \langle (-\ell_4) 4\rangle} \frac{[(-\ell_5) 5]^3}{[5\ell_1]\,[\ell_1 (-\ell_5)]} \\
&= -\frac{1}{2}\frac{\langle 12\rangle^3 \left\langle 3^+ \left|\not\ell_4 \not\ell_5\right| 5^-\right\rangle^3}{\left\langle 2^- \left|\not\ell_3\right| 3^-\right\rangle \left\langle 4^- \left|\not\ell_4 \not\ell_3 \not\ell_1\right| 5^-\right\rangle \left\langle 1^- \left|\not\ell_1 \not\ell_5\right| 4^+\right\rangle}\,.
\end{aligned} \tag{H.585}$$

To get to the last step, we combined spinor products into longer strings using the replacement $|\ell_i\rangle\,[\ell_i| \to \not\ell_i$, but we did not need to use any other properties of the ℓ_i.

3. Use momentum conservation, i.e. $\ell_1 = \ell_4 - p_4 - p_5, \ell_3 = \ell_4 + p_3$ and $\ell_5 = \ell_4 - p_4$, as well as $\ell_i^2 = 0$, and spinor identities to show that

$$d_{12} = \frac{1}{2}\frac{\langle 12\rangle^3 \left\langle 4^- \left|\not\ell_4\right| 3^-\right\rangle^2 [45]^3}{\left\langle 2^- \left|\not\ell_4\right| 3^-\right\rangle \langle 34\rangle [45] \langle 15\rangle \left\langle 4^- \left|\not\ell_4\right| 5^-\right\rangle} \tag{H.586}$$

Solution. Replace

$$\begin{aligned}
\left\langle 3^+ \left|\not\ell_4 \not\ell_5\right| 5^-\right\rangle &\to -\left\langle 4^- \left|\not\ell_4\right| 3^-\right\rangle \langle 45\rangle\,, \\
\left\langle 2^- \left|\not\ell_3\right| 3^-\right\rangle &\to \left\langle 2^- \left|\not\ell_4\right| 3^-\right\rangle\,, \\
\left\langle 4^- \left|\not\ell_4 \not\ell_3 \not\ell_1\right| 5^-\right\rangle &\to \left\langle 4^- \left|\not\ell_4 \not p_3 \left(\not\ell_4 - \not p_4\right)\right| 5^-\right\rangle = -\left\langle 4^- \left|\not\ell_4\right| 3^-\right\rangle \langle 34\rangle [45]\,, \\
\left\langle 1^- \left|\not\ell_1 \not\ell_5\right| 4^+\right\rangle &\to -\langle 15\rangle \left\langle 4^- \left|\not\ell_4\right| 5^-\right\rangle\,.
\end{aligned}$$

4. The explicit analytic expression for one cut loop momentum $\ell_4^\pm$ is given by

$$(\ell_4^\pm)^\mu = \frac{\left\langle 4^\mp \left| \not{p}_5(\not{p}_1+\not{p}_2)\not{p}_3\gamma^\mu \right| 4^\pm \right\rangle}{2\left\langle 4^\mp|\not{p}_5\not{p}_3|4^\pm\right\rangle}\,. \tag{H.587}$$

Show that, with this value of ℓ_4, d_{12} becomes

$$d_{12} = -\frac{1}{2}\frac{\langle 12\rangle^3 s_{34}s_{45}}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \tag{H.588}$$

Solution.

$$(\ell_4^\pm)^\mu = \frac{\left\langle 4^\mp \left| \not{p}_5(\not{p}_1+\not{p}_2)\not{p}_3\gamma^\mu \right| 4^\pm \right\rangle}{2\left\langle 4^\mp|\not{p}_5\not{p}_3|4^\pm\right\rangle} = -\frac{\left\langle 4^\mp \left| \not{p}_5\not{p}_4\not{p}_3\gamma^\mu \right| 4^\pm \right\rangle}{2\left\langle 4^\mp|\not{p}_5\not{p}_3|4^\pm\right\rangle} = -\frac{\left\langle 5^\pm \left| \not{p}_4\not{p}_3\gamma^\mu \right| 4^\pm \right\rangle}{2\left\langle 5^\pm|\not{p}_3|4^\pm\right\rangle}. \tag{H.589}$$

In order to obtain eq. H.588 we need Fierzing in $\ell_4^\mu \propto \langle 3^- |\gamma^\mu| 4^-\rangle$.

H.49 Amplitude of Higgs production from gluon fusion

The coupling of the Higgs boson to the gluons is mediated by a heavy-quark loop. Thus, Higgs production from gluon fusion is a loop-induced process with the leading order contribution shown in fig. H.20. In app. H.25, we considered Higgs production in the Higgs Effective Field Theory, where the coupling of the Higgs boson to the gluons is taken in the limit of an infinite heavy-quark mass. In the following we investigate Higgs production from gluon fusion with full heavy-quark mass dependence.

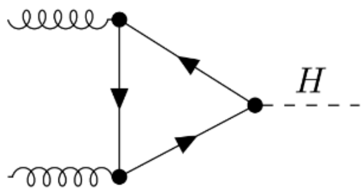


Figure H.20: LO Higgs production in gluon fusion. The process where the gluon-legs are crossed does contribute the same.

The amplitude reads

$$\begin{aligned} i\mathcal{A} = &-2(i)^3(-ig_s)^2\epsilon_{\mu_1}(p_1)\epsilon_{\mu_2}(p_2)\,\mathrm{Tr}\big(t^a t^b\big)\left(\frac{-im_T}{v}\right) \\ &\times \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{t^{\mu_1\mu_2}}{(k^2-m_T^2)\,((k-p_1)^2-m_T^2)((k+p_2)^2-m_T^2)}\,, \end{aligned} \tag{H.590}$$

where

$$t^{\mu\nu} = \mathrm{Tr}\left[(\not{k} + m_T)\gamma^\mu(\not{k} - \not{p}_1 + m_T)(\not{k} + \not{p}_2 + m_T)\gamma^\nu\right] , \tag{H.591}$$

and where the coupling of the Higgs boson to the quarks is proportional to the quark mass m_T, v is the vacuum expectation value, and $(p_1 + p_2)^2 = m_H^2$.

i) Perform the numerator algebra.

Solution. After performing the trace the amplitude reads

$$\mathcal{A} = \epsilon_{\mu_1}(p_1)\epsilon_{\mu_2}(p_2) \underbrace{\frac{2g_s^2 m_T^2 \delta_{a_2,a_1}}{v} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{g_{\mu_1\mu_2}\left(m_H^2 + 2k^2 - 2m_T^2\right) - 8k_{\mu_1}k_{\mu_2} - 2(p_1)_{\mu_2}(p_2)_{\mu_1}}{(k^2 - m_T^2)\left((k-p_1)^2 - m_T^2\right)((k+p_2)^2 - m_T^2)}}_{=:I_{\mu_1\mu_2}} . \tag{H.592}$$

ii) The amplitude you found has still contractions between polarisation vectors and loop-momenta. However, we can bring it in the form,

$$\mathcal{A} = (\epsilon^{\mu_1}(p_1)\epsilon^{\mu_2}(p_2)T_{\mu_1\mu_2})A\,, \tag{H.593}$$

where A does not involve any polarisation vectors and

$$T_{\mu_1\mu_2} = -\frac{1}{2}m_H^2 g_{\mu_1\mu_2} + (p_2)_{\mu_1}(p_1)_{\mu_2}\,, \tag{H.594}$$

is a gauge invariant tensor structure. To obtain A, we apply the projector,

$$P = \epsilon^{\rho_1}(p_1)\epsilon^{\rho_2}(p_2)P_{\rho_1\rho_2}, \qquad \text{with} \qquad P_{\rho_1\rho_2} = \frac{4}{(d-2)m_H^4}T_{\rho_1,\rho_2}\,, \tag{H.595}$$

to the amplitude.

Compute

$$(P, \mathcal{A}) = \sum_{\text{pol.}} P^* \mathcal{A} = A\,. \tag{H.596}$$

Hint. Use e.g. $\epsilon(p_1, p_2)$ and $\epsilon(p_2, p_1)$ for the polarisations.

Solution. We have

$$\begin{aligned}
&\sum_{\text{pol.}} P^* \epsilon_{\mu_1}(p_1, p_2)\epsilon_{\mu_2}(p_2, p_1) \\
&= \left(2\frac{(p_2)_{\mu_2}(p_1)_{\rho_2} + (p_1)_{\mu_2}(p_2)_{\rho_2}}{m_H^2} - g_{\mu_2,\rho_2}\right)\left(2\frac{(p_2)_{\mu_1}(p_1)_{\rho_1} + (p_1)_{\mu_1}(p_2)_{\rho_1}}{m_H^2} - g_{\mu_1,\rho_1}\right)T^{\rho_1\rho_2} \\
&= -\frac{2g_{\mu_2,\mu_1}}{(d-2)m_H^2} + \frac{4(p_1)_{\mu_2}(p_2)_{\mu_1}}{(d-2)m_H^4} + \frac{4(p_1)_{\mu_1}(p_2)_{\mu_2}}{(d-2)m_H^4}\,,
\end{aligned} \tag{H.597}$$

which means

$$A = \Big(- \frac{2g_{\mu_2,\mu_1}}{(d-2)m_H^2} + \frac{4(p_1)_{\mu_2}(p_2)_{\mu_1}}{(d-2)m_H^4} + \frac{4(p_1)_{\mu_1}(p_2)_{\mu_2}}{(d-2)m_H^4} \Big) I^{\mu_1\mu_2}$$
$$= -\frac{4g_s^2 m_T^2 \delta_{a_2,a_1}}{(d-2)v m_H^4} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(m_H^2\left((d-2)\left(m_H^2 - 2m_T^2\right) + 2(d-6)k^2\right) + 32(k\cdot p_1)(k\cdot p_2))}{(k^2 - m_T^2)\left((k-p_1)^2 - m_T^2\right)((k+p_2)^2 - m_T^2)}, \tag{H.598}$$

where $I_{\mu\nu}$ is given above.

iii) We take the scalar three-point topology as

$$i_{n_1,n_2,n_3} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3}}$$
$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m_T^2)^{n_1}((k-p_1)^2 - m_T^2)^{n_2}((k+p_2)^2 - m_T^2)^{n_3}}. \tag{H.599}$$

Express A in terms of scalar integrals of the family i_{n_1,n_2,n_3} only.

Solution. We express all scalar products in terms of propagators,

$$(k\cdot k) = D_1 + m_T^2 \qquad (k\cdot p_1) = \frac{1}{2}(D_1 - D_2) \qquad (k\cdot p_2) = \frac{1}{2}(D_3 - D_1)\,. \tag{H.600}$$

So we have

$$A = -\frac{4g_s^2 m_T^2 \delta_{a_2,a_1}}{(d-2)v m_H^4} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(m_H^2\left((d-2)\left(m_H^2 - 2m_T^2\right) + 2(d-6)k^2\right) + 32(k\cdot p_1)(k\cdot p_2))}{(k^2 - m_T^2)\left((k-p_1)^2 - m_T^2\right)((k+p_2)^2 - m_T^2)}$$
$$= \frac{4g_s^2 m_T^2 \delta_{a_2,a_1}}{(d-2)v m_H^4} \Big[m_H^2 \left(i_{1,1,1}\left(8m_T^2 - (d-2)m_H^2\right) - 2(d-6) i_{0,1,1}\right) + 8(i_{-1,1,1} - i_{0,0,1} - i_{0,1,0} + i_{1,0,0}) \Big]\,. \tag{H.601}$$

iv) Draw all graphs for the scalar (sub)-topologies, assuming $n_i \geq 0$. If $n_i = 0$, the corresponding propagator is shrunk to a point. How many unique scalar topologies are there?

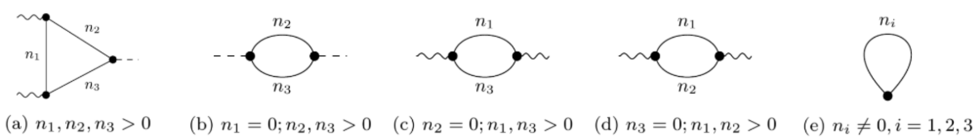


Figure H.21: In the diagram the dashed line corresponds to the Higgs, wavy lines denote massless and continuous straight lines massive propagators.

Solution. We have the scalar topologies shown in H.21. Note in particular, that the cases with $n_2 = 0, n_1, n_3 > 0$ and $n_3 = 0; n_1, n_2 > 0$ are tadpoles as well, since the external momentum squared is 0. So the only unique scalar topologies are the triangle, the massive bubble with incoming momentum $(p_1 + p_2)$ and the tadpoles without any incoming momentum.

v) In the exercise class we discussed IBP identities. How many IBP relations can be derived?

Solution. We have three,

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \partial_{k_\mu} \frac{(p_1)_\mu}{(k^2 - m_T^2)^{n_1}((k-p_1)^2 - m_T^2)^{n_2}((k+p_2)^2 - m_T^2)^{n_3}} = 0\,, \tag{H.602}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \partial_{k_\mu} \frac{(p_2)_\mu}{(k^2 - m_T^2)^{n_1}((k-p_1)^2 - m_T^2)^{n_2}((k+p_2)^2 - m_T^2)^{n_3}} = 0\,, \tag{H.603}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \partial_{k_\mu} \frac{k_\mu}{(k^2 - m_T^2)^{n_1}((k-p_1)^2 - m_T^2)^{n_2}((k+p_2)^2 - m_T^2)^{n_3}} = 0\,. \tag{H.604}$$

vi) Derive the IBP-relations for the scalar topologies: $i_{n_1,n_2,0}$ and $i_{n_1,0,n_3}$ with $n_i \geq 0$.

Solution. The point is to notice, that these topologies are equivalent to tadpoles,

$$i_{n_1,n_2,0} = i_{0,0,n_1+n_2} \qquad\qquad i_{n_1,0,n_3} = i_{0,0,n_1+n_3}\,, \tag{H.605}$$

for which the IBP relation is trivial and was shown in the exercise class,

$$i_{0,0,n} = \frac{d - 2(n-1)}{2(n-1)m_T^2} i_{0,0,n-1}\,. \tag{H.606}$$

vii) We choose the Master Integrals (MIs) $\{i_{0,0,1}, i_{0,1,1}, i_{1,1,1}\}$. Use

$$i_{-1,1,1} = i_{0,0,1} - \frac{1}{2} i_{0,1,1} m_H^2\,, \tag{H.607}$$

and reduce the complete amplitude to be expressed completely in terms of the MIs.

Solution. We found in the previous exercise,

$$\begin{aligned} A = \frac{4 g_s^2 m_T^2 \delta_{a_2,a_1}}{(d-2) v m_H^4} \Big[& m_H^2 \left(i_{1,1,1} \left(8 m_T^2 - (d-2) m_H^2 \right) - 2(d-6) i_{0,1,1} \right) \\ & + 8 (i_{-1,1,1} - i_{0,0,1} - i_{0,1,0} + i_{1,0,0}) \Big]\,. \end{aligned} \tag{H.608}$$

Using the symmetries,

$$i_{0,1,0} = i_{0,0,1}, \qquad\qquad i_{1,0,0} = i_{0,0,1}\,, \tag{H.609}$$

and the provided IBP relation,

$$i_{-1,1,1} = i_{0,0,1} - \frac{1}{2} i_{0,1,1} m_H^2\,, \tag{H.610}$$

we get

$$A = \frac{4 g_s^2 m_T^2 \delta_{a_2,a_1}}{(d-2) v m_H^2} \left(i_{1,1,1} \left(8 m_T^2 - (d-2) m_H^2 \right) - 2(d-4) i_{0,1,1} \right)\,. \tag{H.611}$$

So the amplitude is very compact and only depends on a scalar triangle and a scalar bubble.

H.50 Properties of differential equations

In app. H.51, we compute the Feynman integrals with the method of differential equations. Here we show some important properties of the differential equations for Feynman integrals, which allow for important checks in practical computation.

i) **Basis Transformations:** Suppose a vector of Master Integrals fulfills the differential equation,

$$\partial_x \vec{I} = A_x(x, \epsilon)\vec{I}\,. \tag{H.612}$$

Show that the integrals $\vec{J}$ which are related to $\vec{I}$ with the transformation,

$$\vec{J} = T(x, \epsilon)\vec{I}\,, \tag{H.613}$$

fulfill the differential equation,

$$\partial_x \vec{J} = \tilde{A}_x(x, \epsilon)\vec{J} = \big((\partial_x T)T^{-1} + T A_x T^{-1}\big)\vec{J}\,. \tag{H.614}$$

Solution.

$$\begin{aligned}\partial_x \vec{J} &= \partial_x (T\vec{I}) \\ &= ((\partial_x T) + T A_x)\vec{I} \\ &= ((\partial_x T) + T A_x)T^{-1}\vec{J}\,.\end{aligned} \tag{H.615}$$

ii) **Integrability Condition:** Suppose a vector of integrals depends on n kinematic invariants x_i (masses and Mandelstam variables) and fulfills the partial differential equations,

$$\partial_{x_i} \vec{I} = \partial_i \vec{I} = A_i(x_1, \ldots, x_n, \epsilon)\vec{I}, \qquad \forall\ i = 1, \ldots, n\,. \tag{H.616}$$

Show that

$$\partial_i A_j - \partial_j A_i - [A_i, A_j] = 0\,, \tag{H.617}$$

where $[A, B]$ denotes the matrix commutator.

Hint. Use either $d^2\vec{I} = 0$ or $(\partial_i\partial_j - \partial_j\partial_i)\vec{I} = 0$.

Solution. We have

$$
\begin{aligned}
d^2 I &= d\big(\sum_{i=1}^{n} \partial_i I dx^i\big) \\
&= \sum_{i=1}^{n} d(\partial_i I) \wedge dx^i \\
&= \sum_{i,j=1}^{n} (\partial_j \partial_i I) dx^j \wedge dx^i \\
&= \sum_{i} \sum_{j<i} (\partial_j \partial_i - \partial_i \partial_j) I dx^j \wedge dx^i \\
&= (\partial_j (A_i I) + \partial_i (A_j I)) dx^j \wedge dx^i \\
&= (\partial_j A_i + A_i A_j - \partial_i A_j - A_j A_i) I dx^j \wedge dx^i \\
&= 0\,. \qquad \text{(H.618)} \\
&\Rightarrow \partial_j A_i - \partial_i A_j + [A_i, A_j] = 0\,. \qquad \text{(H.619)}
\end{aligned}
$$

iii) **Scaling Relation:** Suppose a vector of k basis integrals depends on n kinematic invariants x_i (masses and Mandelstam variables) and fulfills the partial differential equations,

$$
\partial_{x_i} \vec{I} = \partial_i \vec{I} = A_i(x_1, \ldots, x_n, \epsilon) \vec{I} \qquad \forall\ i = 1, \ldots, n\,. \tag{H.620}
$$

Show that

$$
\sum_{i=1}^{n} x_i A_i = \mathrm{diag}\Big(\frac{[I_1]}{2}, \ldots, \frac{[I_k]}{2}\Big)\,, \tag{H.621}
$$

where $[I_j]$ denotes the energy dimension of the integral I_j.

Hint. Under rescaling $x_i \to \lambda x_i$ the integral $I_i(x_1, \ldots, x_n)$ will transform as $I_i(\lambda x_1, \ldots, \lambda x_n) = \lambda^{[I_i]/2} I_i(x_1, \ldots, x_n)$.

Solution. We denote $x'_i = \lambda x_i$ and consider

$$
\begin{aligned}
\frac{d}{d\lambda} \vec{I}(\lambda x_1, \ldots, \lambda x_n) &= \sum_{i=1}^{n} \partial_{x'_i} I(x'_1, \ldots, x'_n) \frac{dx'_i}{d\lambda} \\
&= \sum_{i=1}^{n} A_i(x'_1, \ldots, x'_n) I(x'_1, \ldots, x'_n) x_i\,. \qquad \text{(H.622)}
\end{aligned}
$$

On the other hand we know the scaling of the integrals, since they are homogeneous in the mass dimension,

$$\frac{d}{d\lambda}\vec{I}(\lambda x_1,\ldots,\lambda x_n) = \frac{d}{d\lambda}\mathrm{diag}(\lambda^{\frac{[I_1]}{2}},\ldots,\lambda^{\frac{[I_k]}{2}})I(x_1,\ldots,x_n)$$
$$= \mathrm{diag}(\frac{[I_1]}{2},\ldots,\frac{[I_k]}{2})\frac{\mathrm{diag}(\lambda^{\frac{[I_1]}{2}},\ldots,\lambda^{\frac{[I_k]}{2}})}{\lambda}I(x_1,\ldots,x_n)\,, \tag{H.623}$$

so we have

$$\sum_{i=1}^{n} A_i(x_1',\ldots,x_n')I(x_1',\ldots,x_n')x_i = \mathrm{diag}(\frac{[I_1]}{2},\ldots,\frac{[I_k]}{2})\frac{\mathrm{diag}(\lambda^{\frac{[I_1]}{2}},\ldots,\lambda^{\frac{[I_k]}{2}})}{\lambda}I(x_1,\ldots,x_n) \quad \forall\ \lambda\,, \tag{H.624}$$

and in particular we can set $\lambda = 1$ such that $x_i' = x_i$,

$$\sum_{i=1}^{n} A_i(x_1,\ldots,x_n)x_iI(x_1,\ldots,x_n) = \mathrm{diag}(\frac{[I_1]}{2},\ldots,\frac{[I_k]}{2})I(x_1,\ldots,x_n)\,, \tag{H.625}$$

which derives the scaling relation.

H.51 The canonical differential equation for $gg \to H$

*(The following exercises will involve some matrix multiplication which can become tedious. You can find the relevant matrices in electronic form in the file **reference_matrices.nb**)*

We choose the basis integrals $\vec{i} = (i_{0,0,1}, i_{0,1,1}, i_{1,1,1})^T$ for the amplitude computation. We now want to derive the differential equation for these integrals and find a canonical basis, in which the full dependence of the dimensional regulator is factorized.

i) Derive the differential for $\vec{i} = (i_{0,0,1}, i_{0,1,1}, i_{1,1,1})$ w.r.t. m_H^2 and m_T^2. The relevant IBPs are

$$i_{-1,2,1} = \frac{i_{0,1,1}\left((d-2)m_H^2 - 4m_T^2\right) - 2(d-2)i_{0,0,1}}{2\left(m_H^2 - 4m_T^2\right)}\,, \tag{H.626}$$

$$i_{0,2,1} = \frac{2(d-3)i_{0,1,1}m_T^2 - (d-2)i_{0,0,1}}{8m_T^4 - 2m_H^2 m_T^2}\,, \tag{H.627}$$

$$i_{0,0,2} = \frac{(d-2)i_{0,0,1}}{2m_T^2}\,, \tag{H.628}$$

$$i_{0,1,2} = \frac{(d-2)i_{0,0,1}}{2m_T^2\left(m_H^2 - 4m_T^2\right)} - \frac{(d-3)i_{0,1,1}}{m_H^2 - 4m_T^2}\,, \tag{H.629}$$

$$i_{0,2,1} = \frac{(d-2)i_{0,0,1}}{2m_T^2\left(m_H^2 - 4m_T^2\right)} - \frac{(d-3)i_{0,1,1}}{m_H^2 - 4m_T^2}\,, \tag{H.630}$$

$$i_{1,1,2} = \frac{(d-2)i_{0,0,1}\left(dm_H^2 - 4dm_T^2 - 4m_H^2 + 12m_T^2\right)}{4m_H^2 m_T^4\left(m_H^2 - 4m_T^2\right)} + \frac{2(d-3)i_{0,1,1}}{m_H^2\left(m_H^2 - 4m_T^2\right)}\,, \tag{H.631}$$

$$i_{1,2,1} = \frac{(d-2)i_{0,0,1}\left(dm_H^2 - 4dm_T^2 - 4m_H^2 + 12m_T^2\right)}{4m_H^2 m_T^4\left(m_H^2 - 4m_T^2\right)} + \frac{2(d-3)i_{0,1,1}}{m_H^2\left(m_H^2 - 4m_T^2\right)}\,, \tag{H.632}$$

$$i_{2,1,1} = -\frac{(d-3)(d-2)i_{0,0,1}}{2m_H^2 m_T^4} + \frac{(d-3)i_{0,1,1}}{m_H^2 m_T^2} + \frac{(d-4)i_{1,1,1}}{2m_T^2}\,. \tag{H.633}$$

Solution. We have

$$p_{1,\mu}\partial_{p_{1\mu}} = p_{1,\mu}(\partial_{p_{1\mu}}(2p_1\cdot p_2))\partial_{m_H^2} = m_H^2\partial_{m_H^2}\,, \tag{H.634}$$

which shows

$$\begin{aligned}
\partial_{m_H^2} i_{n_1,n_2,n_3} &= \frac{1}{m_H^2}p_{1,\mu}\partial_{p_{1\mu}} i_{n_1,n_2,n_3} \\
&= \frac{1}{m_H^2}(-n_2)i_{n_1,n_2+1,n_3}2(-p_1\cdot k) \\
&= \frac{1}{m_H^2}(-n_2)i_{n_1,n_2+1,n_3}2\frac{1}{2}(D_2 - D_1) \\
&= (-n_2)\frac{1}{m_H^2}(i_{n_1,n_2,n_3} - i_{n_1-1,n_2+1,n_3})\,,
\end{aligned} \tag{H.635}$$

and so

$$\frac{\partial}{\partial m_H^2}\vec{i} = \begin{pmatrix} 0 \\ \frac{i_{-1,2,1}-i_{0,1,1}}{m_H^2} \\ \frac{i_{0,2,1}-i_{1,1,1}}{m_H^2} \end{pmatrix}, \tag{H.636}$$

$$= \begin{pmatrix} 0 \\ \frac{i_{0,1,1}\left((d-4)m_H^2+4m_T^2\right)-2(d-2)i_{0,0,1}}{2m_H^2\left(m_H^2-4m_T^2\right)} \\ \frac{2m_T^2\left(-(d-3)i_{0,1,1}-i_{1,1,1}\left(m_H^2-4m_T^2\right)\right)+(d-2)i_{0,0,1}}{2m_H^2m_T^2\left(m_H^2-4m_T^2\right)} \end{pmatrix}, \tag{H.637}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \frac{2-d}{m_H^4-4m_H^2m_T^2} & \frac{(d-4)m_H^2+4m_T^2}{2m_H^2\left(m_H^2-4m_T^2\right)} & 0 \\ \frac{d-2}{2m_H^2m_T^2\left(m_H^2-4m_T^2\right)} & \frac{3-d}{m_H^4-4m_H^2m_T^2} & -\frac{1}{m_H^2} \end{pmatrix}}_{A_{m_H^2}}\vec{i}\,. \tag{H.638}$$

$$\frac{\partial}{\partial m_T^2}\vec{i} = \begin{pmatrix} i_{0,0,2} \\ i_{0,1,2}+i_{0,2,1} \\ i_{1,1,2}+i_{1,2,1}+i_{2,1,1} \end{pmatrix}, \tag{H.639}$$

$$= \begin{pmatrix} \frac{(d-2)i_{0,0,1}}{2m_T^2} \\ \frac{(d-2)i_{0,0,1}-2(d-3)i_{0,1,1}m_T^2}{m_T^2\left(m_H^2-4m_T^2\right)} \\ \frac{m_T^2\left(-(d-4)i_{1,1,1}\left(m_H^2-4m_T^2\right)-2(d-3)i_{0,1,1}\right)+(d-2)i_{0,0,1}}{8m_T^6-2m_H^2m_T^4} \end{pmatrix}, \tag{H.640}$$

$$= \underbrace{\begin{pmatrix} \frac{d-2}{2m_T^2} & 0 & 0 \\ \frac{d-2}{m_T^2\left(m_H^2-4m_T^2\right)} & \frac{6-2d}{m_H^2-4m_T^2} & 0 \\ -\frac{d-2}{2m_T^4\left(m_H^2-4m_T^2\right)} & \frac{d-3}{m_T^2\left(m_H^2-4m_T^2\right)} & \frac{d-4}{2m_T^2} \end{pmatrix}}_{A_{m_T^2}}\vec{i}\,. \tag{H.641}$$

ii) In the previous exercise you found

$$\frac{\partial}{\partial m_H^2}\vec{i} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \frac{2-d}{m_H^4-4m_H^2m_T^2} & \frac{(d-4)m_H^2+4m_T^2}{2m_H^2\left(m_H^2-4m_T^2\right)} & 0 \\ \frac{d-2}{2m_H^2m_T^2\left(m_H^2-4m_T^2\right)} & \frac{3-d}{m_H^4-4m_H^2m_T^2} & -\frac{1}{m_H^2} \end{pmatrix}}_{A_{m_H^2}}\vec{i}, \tag{H.642}$$

$$\frac{\partial}{\partial m_T^2}\vec{i} = \underbrace{\begin{pmatrix} \frac{d-2}{2m_T^2} & 0 & 0 \\ \frac{d-2}{m_T^2\left(m_H^2-4m_T^2\right)} & \frac{6-2d}{m_H^2-4m_T^2} & 0 \\ -\frac{d-2}{2m_T^4\left(m_H^2-4m_T^2\right)} & \frac{d-3}{m_T^2\left(m_H^2-4m_T^2\right)} & \frac{d-4}{2m_T^2} \end{pmatrix}}_{A_{m_T^2}}\vec{i}\,. \tag{H.643}$$

Verify the scaling relation and the integrability condition.

We find the scaling relation,

$$m_H^2 A_{m_H^2} + m_T^2 A_{m_T^2} = \begin{pmatrix} \frac{d-2}{2} & 0 & 0 \\ 0 & \frac{d-4}{2} & 0 \\ 0 & 0 & \frac{d-6}{2} \end{pmatrix}, \tag{H.644}$$

which is the expected result. For the integrability condition we need

$$\partial_{m_T^2} A_{m_H^2} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{8(\epsilon-1)}{m_H^2\left(m_H^2-4m_T^2\right)^2} & \frac{2-4\epsilon}{\left(m_H^2-4m_T^2\right)^2} & 0 \\ \frac{(\epsilon-1)\left(m_H^2-8m_T^2\right)}{m_H^2 m_T^4\left(m_H^2-4m_T^2\right)^2} & \frac{8\epsilon-4}{m_H^2\left(m_H^2-4m_T^2\right)^2} & 0 \end{pmatrix}, \tag{H.645}$$

$$\partial_{m_H^2} A_{m_T^2} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{2(\epsilon-1)}{m_T^2\left(m_H^2-4m_T^2\right)^2} & \frac{2-4\epsilon}{\left(m_H^2-4m_T^2\right)^2} & 0 \\ \frac{1-\epsilon}{m_T^4\left(m_H^2-4m_T^2\right)^2} & \frac{2\epsilon-1}{m_T^2\left(m_H^2-4m_T^2\right)^2} & 0 \end{pmatrix}, \tag{H.646}$$

$$[A_{m_T^2}, A_{m_H^2}] = \begin{pmatrix} 0 & 0 & 0 \\ \frac{2-2\epsilon}{m_H^4 m_T^2-4m_H^2 m_T^4} & 0 & 0 \\ \frac{2(\epsilon-1)}{m_H^2 m_T^4\left(m_H^2-4m_T^2\right)} & \frac{1-2\epsilon}{m_H^4 m_T^2-4m_H^2 m_T^4} & 0 \end{pmatrix}, \tag{H.647}$$

and we find

$$\begin{aligned} \partial_{m_T^2} A_{m_H^2} - \partial_{m_H^2} A_{m_T^2} - [A_{m_T^2}, A_{m_H^2}] &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{2-2\epsilon}{m_H^4 m_T^2-4m_H^2 m_T^4} & 0 & 0 \\ \frac{2(\epsilon-1)}{m_H^2 m_T^4\left(m_H^2-4m_T^2\right)} & \frac{1-2\epsilon}{m_H^4 m_T^2-4m_H^2 m_T^4} & 0 \end{pmatrix} - [A_{m_T^2}, A_{m_H^2}] \\ &= 0, \end{aligned} \tag{H.648}$$

as expected.

iii) The basis integrals $\vec{i}$ are not a optimal basis candidate, even though for one-loop the basis of integrals is not too crucial. A better basis is given by

$$\vec{j} = \begin{pmatrix} \epsilon i_{0,0,2} \\ \epsilon i_{0,2,1} \\ \epsilon^2 i_{1,1,1} \end{pmatrix}, \tag{H.649}$$

which relates to the basis $\vec{i}$ with transformation matrix $T(m_T, m_H, \epsilon)$

$$\vec{j} = T(m_T, m_H, \epsilon)\vec{i} = \begin{pmatrix} \frac{(2-2\epsilon)\epsilon}{2m_T^2} & 0 & 0 \\ \frac{(2-2\epsilon)\epsilon}{2m_T^2\left(m_H^2-4m_T^2\right)} & -\frac{(1-2\epsilon)\epsilon}{m_H^2-4m_T^2} & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}\vec{i}\,. \tag{H.650}$$

The basis $\vec{j}$ fulfill the differential equation,

$$\partial_{s_i}\vec{j} = \tilde{A}_{s_i}\vec{j}, \qquad s_i \in \{m_H^2, m_T^2\}\,. \tag{H.651}$$

Compute the matrices $\tilde{A}_{s_i}$.

Solution. The general transformation is

$$\tilde{A}_{s_i} = (\partial_{s_i}T)T^{-1} + TA_{s_i}T^{-1}\,, \tag{H.652}$$

where $\partial_{s_i}\vec{i} = A_{s_i}\vec{i}$. So we find

$$\tilde{A}_{m_T^2} = \begin{pmatrix} -\frac{\epsilon}{m_T^2} & 0 & 0 \\ \frac{\epsilon}{4m_T^4 - m_H^2 m_T^2} & \frac{4\epsilon}{m_H^2 - 4m_T^2} + \frac{2}{m_H^2 - 4m_T^2} & 0 \\ 0 & -\frac{\epsilon}{m_T^2} & -\frac{\epsilon}{m_T^2} \end{pmatrix}, \tag{H.653}$$

$$\tilde{A}_{m_H^2} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\epsilon}{m_H^4 - 4m_H^2 m_T^2} & \frac{\epsilon}{4m_T^2 - m_H^2} - \frac{m_H^2 - 2m_T^2}{m_H^4 - 4m_H^2 m_T^2} & 0 \\ 0 & \frac{\epsilon}{m_H^2} & -\frac{1}{m_H^2} \end{pmatrix}. \tag{H.654}$$

iv) The matrices $\tilde{A}$ are still not in ϵ-factorized form, since the differential equation reads

$$\partial_{s_i}\vec{j} = \left(\tilde{A}^{(0)}_{s_i}(m_H, m_T) + \epsilon\tilde{A}^{(1)}_{s_i}(m_H, m_T)\right)\vec{j}, \qquad s_i \in \{m_H^2, m_T^2\}\,. \tag{H.655}$$

In order to bring the differential equation into canonical form, we need to do one last transformation. To find that transformation, we work on the *maximal cut* of a scalar topology, for which all sub-topologies vanish and the differential equation becomes the homogeneous differential equation,

$$\partial_{s_i}\vec{j}|_{\text{max. cut}} = \left(\tilde{A}^{(0)}_{s_i}(m_H, m_T)|_{\text{max. cut}} + \epsilon\tilde{A}^{(1)}_{s_i}(m_H, m_T)|_{\text{max. cut}}\right)\vec{j}|_{\text{max. cut}}\,. \tag{H.656}$$

where $[\tilde{A}_{s_i}|_{\text{max. cut}}]_{ij} = \delta_{ij}[\tilde{A}_{s_i}]_{ij}$ contains only the main diagonal of $\tilde{A}_{s_i}$. We want to determine a transformation,

$$\vec{c} = T_2(m_T, m_H)\vec{j}\,, \tag{H.657}$$

such that

$$\partial_{s_i}\vec{c}|_{\text{max. cut}} = \epsilon\hat{A}_{s_i}(m_H, m_T)|_{\text{max. cut}}\vec{c}|_{\text{max. cut}}\,. \tag{H.658}$$

T_2 is independent of the dimensional regulator ϵ.

- Show that the transformation T_2 fulfills the differential equation,

$$\partial_{s_i} T_2 = -T_2 \tilde{A}^{(0)}_{s_i}(m_H, m_T)|_{\text{max. cut}} \tag{H.659}$$

or equivalently its inverse fulfills

$$\partial_{s_i} T_2^{-1} = \tilde{A}^{(0)}_{s_i}(m_H, m_T)|_{\text{max. cut}} T_2^{-1}\,. \tag{H.660}$$

Solution. We have

$$\partial_{s_i} \vec{j} = \left(\tilde{A}^{(0)}_{s_i}(m_H, m_T) + \epsilon \tilde{A}^{(1)}_{s_i}(m_H, m_T)\right) \vec{j}, \qquad s_i \in \{m_H^2, m_T^2\}\,, \tag{H.661}$$

so the differential equation for $\vec{c}$ on the maximal cut reads

$$\partial_{s_i} \vec{c}|_{\text{max. cut}} = \partial_{s_i}(T_2 \vec{j})|_{\text{max. cut}} = \left(\partial_{s_i} T_2 + T_2 \tilde{A}^{(0)}_{s_i}|_{\text{max. cut}} + \epsilon T_2 \tilde{A}^{(1)}_{s_i}|_{\text{max. cut}}\right) T_2^{-1} \vec{c}|_{\text{max. cut}}\,. \tag{H.662}$$

Our requirement that this differential equation is in canonical form yields directly

$$\partial_{s_i} T_2 + T_2 \tilde{A}^{(0)}_{s_i}|_{\text{max. cut}} = 0\,, \tag{H.663}$$

or by using $\partial_{s_i}(TT^{-1}) = (\partial_{s_i} T) + T(\partial_{s_i} T^{-1}) = 0$ the analogous one for the inverse.

- Solve the above system of partial differential equations for T_2.

Hint. Make use of the fact that everything is diagonal. Start by solving the partial differential equation (PDE) in e.g. m_T^2, plug the solution in the PDE for m_H^2 and solve it.

Solution. We will solve the differential equation for T_2^{-1}. We start by solving the partial differential equation in m_T^2. So we have to solve

$$\partial_{m_T^2} T_2^{-1} = \tilde{A}^{(0)}_{m_T^2}(m_H, m_T)|_{\text{max. cut}} T_2^{-1}\,, \tag{H.664}$$

which immediately tells us

$$T_2^{-1} = \underbrace{\exp\Big(\int \tilde{A}^{(0)}_{m_T^2}(m_H, m_T)|_{\text{max. cut}} \mathrm{d}m_T^2\Big)}_{\hat{T}_{m_T}(m_T, m_H)} \hat{T}_{m_H}(m_H^2)\,, \tag{H.665}$$

with

$$\hat{T}_{m_T}(m_H, m_T) = \exp\Big(\int \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{m_H^2 - 4m_T^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{d}m_T^2\Big)$$

$$= \exp\Big(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}\log\left(4m_T^2 - m_H^2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}\Big)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4m_T^2 - m_H^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{H.666}$$

with yet to be determined boundary constant $\hat{T}_{m_H}(m_H^2)$. We will from now on suppress the restriction $|_{\text{max. cut}}$, but it is understood, that sub-topolgies are set to zero. To determine $\hat{T}_{m_H}(m_H^2)$ we use the differential equation w.r.t to m_H^2,

$$\partial_{m_H^2} T_2^{-1} = \tilde{A}^{(0)}_{m_H^2}(m_H, m_T) T_2^{-1} \tag{H.667}$$

$$\Leftrightarrow (\partial_{m_H^2}\hat{T}_{m_T})\hat{T}_{m_H} + \hat{T}_{m_T}\partial_{m_H^2}\hat{T}_{m_H} = \tilde{A}^{(0)}_{m_H^2}\hat{T}_{m_T}\hat{T}_{m_H} \tag{H.668}$$

$$\Leftrightarrow \partial_{m_H^2}\hat{T}_{m_H} = \Big(\hat{T}_{m_T}^{-1}\tilde{A}^{(0)}_{m_H^2}\hat{T}_{m_T} - \hat{T}_{m_T}^{-1}\partial_{m_H^2}\hat{T}_{m_T}\Big)\hat{T}_{m_H} \tag{H.669}$$

$$= \Big(\tilde{A}^{(0)}_{m_H^2} - \hat{T}_{m_T}^{-1}\partial_{m_H^2}\hat{T}_{m_T}\Big)\hat{T}_{m_H}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2m_H^2} & 0 \\ 0 & 0 & -\frac{1}{m_H^2} \end{pmatrix}\hat{T}_{m_H}, \tag{H.670}$$

where we made use of the fact that $\tilde{A}^{(0)}_{m_H^2}$ and $\hat{T}_{m_T}$ are diagonal. So

$$\hat{T}_{m_H} = \exp\Big(\int \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2m_H^2} & 0 \\ 0 & 0 & -\frac{1}{m_H^2} \end{pmatrix} \mathrm{d}m_H^2\Big) C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{m_H^2}} & 0 \\ 0 & 0 & \frac{1}{m_H^2} \end{pmatrix} C, \tag{H.671}$$

and we can set the constant matrix C to be $\mathbb{1}$ (other nonsingular choices are of course fine as well, in particular in the next exercise we will use C_{22} to be i to make everything well defined in the Euclidean region). Putting everything together we have

$$T_2 = (\hat{T}_{m_T}\hat{T}_{m_H})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{m_H^2}\sqrt{4m_T^2 - m_H^2} & 0 \\ 0 & 0 & m_H^2 \end{pmatrix}. \tag{H.672}$$

- Derive the full differential equations for $\vec{c} = T_2(m_T, m_H)\vec{j}$ (not only on the maximal cut). Do we need other transformations or is everything already in canonical form?

  Solution. We have

$$\partial_{m_T^2}\vec{c} = ((\partial_{m_T^2}T_2) + T_2\tilde{A}_{m_T})T_2^{-1}\vec{c} = \epsilon \begin{pmatrix} -\frac{1}{m_T^2} & 0 & 0 \\ \frac{\sqrt{m_H^2}}{m_T^2\sqrt{4m_T^2 - m_H^2}} & \frac{4}{m_H^2 - 4m_T^2} & 0 \\ 0 & -\frac{\sqrt{m_H^2}}{m_T^2\sqrt{4m_T^2-m_H^2}} & -\frac{1}{m_T^2} \end{pmatrix} \vec{c}, \tag{H.673}$$

  and

$$\partial_{m_H^2}\vec{c} = ((\partial_{m_H^2}T_2) + T_2\tilde{A}_{m_H})T_2^{-1}\vec{c} = \epsilon \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{m_H^2}\sqrt{4m_T^2-m_H^2}} & \frac{1}{4m_T^2 - m_H^2} & 0 \\ 0 & \frac{1}{\sqrt{m_H^2}\sqrt{4m_T^2-m_H^2}} & 0 \end{pmatrix} \vec{c}. \tag{H.674}$$

  So indeed, everything is already in canonical form and we will not need other transformations (it is due to our choice of the basis $\vec{j}$, which was handcrafted, such that no other transformations will be necessary).

  In general, that will not be the case. Then another step will be required: fixing the subtopologies. To do that, you would work your way up starting from the simplest (least denominators) topologies. Since the complete main diagonal will always be in canonical form (due to our transformation T_2) at this point, only the off-diagonal matrix elements would need to be fixed. For our particular example, the first step would be to define a new basis integral $\tilde{c}_2 = c_2 + f(m_H, m_t)c_1$, such that the matrix element A_{21} will become canonical (you would again solve a differential equation for f). The new basis at this point would be $\{c_1, \tilde{c}_2, c_3\}$. Then you would continue with the triangle. First you would define $\tilde{c}_3 = c_3 + g(m_H, m_t)\tilde{c}_2$, to fix the entry A_{32} by determining g. The new basis would be $\{c_1, \tilde{c}_2, \tilde{c}_3\}$ and only A_{31} would not be ϵ-factorized. This would be a last transformation $\{c_1, \tilde{c}_2, \tilde{c}_3\} \to \{c_1, \tilde{c}_2, \tilde{\tilde{c}}_3\}$, where $\tilde{\tilde{c}}_3 = \tilde{c}_3 + h(m_H, m_T)c_1$ such that A_{31} becomes canonical. This is in practice (usually) not the bottleneck, since the consecutive transformations are (often) relatively simple.

v) we want to integrate the differential equation and solve it in terms of special functions called harmonic polylogarithms, defined in eq. (2.81). We will use a slightly different normalisation,

$$i_{n_1,n_2,n_3} = \left(m_T^{2\epsilon}\frac{e^{\gamma_E\epsilon}}{i\pi^{d/2}}\right)\int \mathrm{d}^d k \frac{1}{(k^2 - m_T^2)^{n_1}((k-p_1)^2 - m_T^2)^{n_2}((k+p_2)^2 - m_T^2)^{n_3}}\,. \tag{H.675}$$

which gets rid of log π's and γ_E's. In the previous exercise we determined a canonical basis for the scalar topologies of $gg \to H$ as

$$\vec{c} = (\epsilon i_{0,0,2}, \epsilon\sqrt{-m_H^2}\sqrt{4m_T^2 - m_H^2} i_{0,2,1}, \epsilon^2 m_H^2 i_{1,1,1})^T\,, \tag{H.676}$$

with

$$\partial_{m_T^2}\vec{c} = \epsilon \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \frac{\sqrt{-m_H^2}}{m_T^2\sqrt{4m_T^2-m_H^2}} & \frac{m_H^2}{m_H^2 m_T^2 - 4m_T^4} & 0 \\ 0 & \frac{\sqrt{-m_H^2}}{m_T^2\sqrt{4m_T^2-m_H^2}} & 0 \end{pmatrix}}_{A_{m_T}} \vec{c}\,, \tag{H.677}$$

$$\partial_{m_H^2}\vec{c} = \epsilon \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{-m_H^2}\sqrt{4m_T^2-m_H^2}} & \frac{1}{4m_T^2-m_H^2} & 0 \\ 0 & \frac{1}{\sqrt{-m_H^2}\sqrt{4m_T^2-m_H^2}} & 0 \end{pmatrix}}_{A_{m_H}} \vec{c}\,. \tag{H.678}$$

Note that we made the basis well defined for $m_H^2 < 0$ (in the Euclidean).

We can rewrite a canonical differential equation as a total differential equation,

$$\mathrm{d}\vec{c} = \epsilon \mathrm{d}\tilde{A} \cdot \vec{c} = \epsilon \sum_{s_i \in \{m_T^2, m_H^2\}} (\partial_{s_i}\tilde{A})\ \mathrm{d}s_i\ \vec{c} = A_{m_T}\ \vec{c}\ \mathrm{d}m_T^2 + A_{m_H}\ \vec{c}\ \mathrm{d}m_H^2\,. \tag{H.679}$$

Compute the matrix $\tilde{A}$.

Hint. Use e.g. $A_1 = \int \mathrm{d}m_T^2 A_{m_T}$ and $A_2 = \int \mathrm{d}m_H^2(A_{m_H} - \partial_{m_H^2} A_1)$ such that $\tilde{A} = A_1 + A_2$.

Solution. We compute (e.g. with Mathematica)

$$A_1 = \int \mathrm{d}m_T^2 A_{m_T} \tag{H.680}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \log\left(1 - \frac{\sqrt{4m_T^2-m_H^2}}{\sqrt{-m_H^2}}\right) - \log\left(1 + \frac{\sqrt{4m_T^2-m_H^2}}{\sqrt{-m_H^2}}\right) & \log(m_T^2) - \log(4m_T^2 - m_H^2) & 0 \\ 0 & \log\left(1 - \frac{\sqrt{4m_T^2-m_H^2}}{\sqrt{-m_H^2}}\right) - \log\left(1 + \frac{\sqrt{4m_T^2-m_H^2}}{\sqrt{-m_H^2}}\right) & 0 \end{pmatrix}, \tag{H.681}$$

and verify that already $(A_{m_H} - \partial_{m_H^2} A_1)$=0, so we have $\tilde{A} = A_1$.

vi) Having the matrix $\tilde{A}$ makes variable changes particular simple, since all measure changes will be taken care of. We want to consider the change of variables,

$$m_H^2 = \frac{-(1-x)^2}{x} m_T^2\,, \tag{H.682}$$

or equivalently,

$$x = \lim_{\eta \downarrow 0^+} \frac{\sqrt{4 - (\frac{m_H^2}{m_T^2} + i\eta)} - \sqrt{-(\frac{m_H^2}{m_T^2} + i\eta)}}{\sqrt{4 - (\frac{m_H^2}{m_T^2} + i\eta)} + \sqrt{-(\frac{m_H^2}{m_T^2} + i\eta)}}, \tag{H.683}$$

where Feynman's prescription is denoted by the explicit $+i\eta$.

- Sketch the (complex) variable x as a function of $y = \frac{m_H^2}{m_T^2}$.

  Solution.

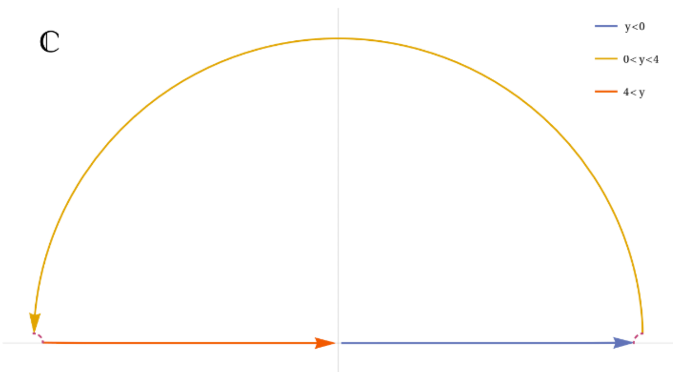


Figure H.22: Representation of the complex variable $x = \frac{\sqrt{4-(y+i\eta)} - \sqrt{-(y+i\eta)}}{\sqrt{4-(y+i\eta)} + \sqrt{-(y+i\eta)}}$ for all kinematic regions. Feynman's prescription is denoted by the explicit $+i\eta$.

We have more explicitly,

$$x + i \lim_{\eta \downarrow 0^+} \eta = \begin{cases} \frac{\sqrt{4-y}-\sqrt{-y}}{\sqrt{4-y}+\sqrt{-y}}; & & y < 0 \\ -e^{i(\phi-\pi)}; & \phi = \arctan\left(\frac{\sqrt{(4-y)y}}{2-y}\right); & 0 < y < 2 \\ -e^{i\phi}; & \phi = \arctan\left(\frac{\sqrt{(4-y)y}}{2-y}\right); & 2 < y < 4 \\ \frac{\sqrt{y-4}-\sqrt{y}}{\sqrt{y-4}+\sqrt{y}}; & & 4 < y \end{cases}. \tag{H.684}$$

The last line indicates that above threshold ($m_H^2 > 4m_T^2$) where $-1 < x < 0$, x has to be evaluated by approaching the negative real axis from the upper half plane. The variable x is shown in fig. H.22.

- Perform the variable change on $\tilde{A}$ and compute $A_x = \partial_x \tilde{A}(x)$.

  Solution. The variable x rationalises the square root,

$$\frac{\sqrt{4m_T^2 - m_H^2}}{\sqrt{-m_H^2}}\Bigg|_{m_H^2 = \frac{-(1-x)^2}{x} m_T^2} = \frac{x+1}{x-1}, \tag{H.685}$$

so that

$$\tilde{A}(m_H(x,m_T),m_T) = \begin{pmatrix} 0 & 0 & 0 \\ -\log(x) & \log(x) - 2\log(x+1) & 0 \\ 0 & -\log(x) & 0 \end{pmatrix}, \tag{H.686}$$

and

$$A_x = \partial_x \tilde{A}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{x} & \frac{1}{x} - \frac{2}{x+1} & 0 \\ 0 & -\frac{1}{x} & 0 \end{pmatrix}. \tag{H.687}$$

vii) As we see, the differential equation,

$$\partial_x \vec{c}(x,m_T) = \epsilon A_x \vec{c} \qquad \Leftrightarrow \qquad \partial_x \vec{c}^{(j)}(x,m_T) = A_x \vec{c}^{(j-1)}, \tag{H.688}$$

where

$$\vec{c} = \sum_{n=0}^{\infty} \epsilon^j \vec{c}^{(j)}, \tag{H.689}$$

and

$$\vec{c} = \begin{pmatrix} \epsilon i_{0,0,2} \\ -\frac{(x-1)(x+1)\epsilon m_T^2}{x} i_{0,2,1} \\ -\frac{(x-1)^2\epsilon^2 m_T^2}{x} i_{1,1,1} \end{pmatrix} \qquad A_x = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{x} & \frac{1}{x} - \frac{2}{x+1} & 0 \\ 0 & -\frac{1}{x} & 0 \end{pmatrix}, \tag{H.690}$$

is completely rationalised.

- Argue why the point $x = 1$ is a good boundary point by investigating the basis.

  Hint. To which value of m_H^2/m_T^2 corresponds $x = 1$? Are the basis integrals finite at this point ?

  Solution. We see that the canonical integrals c_2 and c_3 are multiplied by $x - 1$, so they can have only a non-vanishing value at this point, if either the integrals $i_{0,2,1}$ or $i_{1,1,1}$ are singular for $x = 1$. However, the point $x = 1$ corresponds to the value $m_H^2 = 0$ but all the propagators are massive, so none of the integrals will be singular at this point, since the internal particles cannot be on-shell. To summarise, the only non-vanishing basis integral at $x = 1$ is the tadpole, which is trivial to compute.

- Solve the differential equation in terms of harmonic polylogarithms at least up to order ϵ^2. Harmonic polylogarithms are defined as the iterated integral,

$$H(a_n, a_{n-1}, \ldots, a_1; x) = \int_0^x H(a_{n-1}, \ldots, a_1; t) f(a_n, t) \mathrm{d}t\,,$$

where $a_i \in \{1, 0, -1\}$ and

$$f(1,t) = \frac{1}{1-t}\,, \qquad f(0,t) = \frac{1}{t} \qquad \text{and} \qquad f(-1,t) = \frac{1}{1+t}\,.$$

For the case of all $a_n, \ldots, a_1$ being zero we define

$$H(\underbrace{0,0,\ldots,0}_{n-\text{times}};x) = \frac{1}{n!}\log^n(x)\,.$$

For the boundary condition you will need

$$\begin{aligned} c_1 &= \epsilon i_{0,0,2} \\ &= 1 + \frac{\pi^2\epsilon^2}{12} - \frac{\zeta(3)\epsilon^3}{3} + \frac{\pi^4\epsilon^4}{160} + \left(-\frac{\pi^2\zeta(3)}{36} - \frac{\zeta(5)}{5}\right)\epsilon^5 + O\left(\epsilon^6\right)\,, \end{aligned} \tag{H.691}$$

and

$$H(-1,0;1) = -\frac{\pi^2}{12}\,. \tag{H.692}$$

Solution. We solve order by order in ϵ for the Laurent-coefficient $\vec{c}^{(j)}(x)$ and use as the boundary point $x = 1$, with the boundary value,

$$\vec{c}^{(j)}(x=1) = \left(c_1^{(j)}, 0, 0\right)^T. \tag{H.693}$$

At order ϵ^0 we have trivially

$$\vec{c}^{(0)}(x) = \left(c_1^{(0)}, 0, 0\right)^T = (1,0,0)^T\,. \tag{H.694}$$

For order ϵ we have to solve

$$\partial_x \vec{c}^{(1)} = A_x c^{(0)} = \begin{pmatrix} 0 \\ -\frac{1}{x} \\ 0 \end{pmatrix} \qquad \Leftrightarrow \qquad \vec{c}^{(1)}(x) = \begin{pmatrix} 0 \\ -H(0;x) \\ 0 \end{pmatrix} + \vec{c_0}\,, \tag{H.695}$$

and

$$c^{(1)}(x)|_{x=1} = \vec{c_0} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\,, \tag{H.696}$$

determines the boundary constant $\vec{c_0} = 0$.

For order ϵ^2 we have to solve

$$\partial_x \vec{c}^{(2)} = A_x c^{(1)} = \begin{pmatrix} 0 \\ -\left(\frac{1}{x} - \frac{2}{x+1}\right) H(0;x) \\ \frac{H(0,x)}{x} \end{pmatrix}$$

$$\Rightarrow \vec{c}^{(2)}(x) = \begin{pmatrix} 0 \\ 2H(-1,0;x) - H(0,0;x) \\ H(0,0;x) \end{pmatrix} + \vec{c_0}\,, \tag{H.697}$$

and

$$c^{(2)}(x)|_{x=1} = \begin{pmatrix} 0 \\ -\frac{\pi^2}{6} \\ 0 \end{pmatrix} + \vec{c_0} \stackrel{!}{=} \begin{pmatrix} \frac{\pi^2}{12} \\ 0 \\ 0 \end{pmatrix}\,, \tag{H.698}$$

determines the boundary constant $\vec{c_0} = \left(\frac{\pi^2}{12}, \frac{\pi^2}{6}, 0\right)^T$. Of course higher orders are trivial to obtain.

H.52 Multiple polylogarithms in mathematics

In the physics literature, the MPL's are often defined as the iterated integral,

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)\,. \tag{H.699}$$

In the mathematical literature, the iterated integral is often defined as (notice the index ordering and the base-point),

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0}^{a_{n+1}} \frac{\mathrm{d}t}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t)\,.$$

Express $I(a_0; a_1, a_2; a_3)$ and $I(a_0; a_1, a_2, a_3; a_4)$ in terms of the MPLs (H.699).

Solution. Let us start with

$$I(a_0; a_1, a_2; a_3) = \int_{a_0}^{a_3} \frac{\mathrm{d}t}{t - a_2} I(a_0, a_1; t)\,. \tag{H.700}$$

In sec. 2.6.1, we showed that

$$I(a_0; a_1; t) = G(a_1; t) - G(a_1; a_0)\,,$$

so

$$I(a_0; a_1, a_2; a_3) = \left[\int_0^{a_3} - \int_0^{a_0}\right] \left(G(a_1; t) - G(a_1; a_0)\right) \frac{\mathrm{d}t}{t - a_2}$$
$$= G(a_2, a_1; a_3) - G(a_2, a_1; a_0) - G(a_1; a_0) \left[G(a_2; a_3) - G(a_2; a_0)\right] \,. \tag{H.701}$$

For the weight-three integral we have

$$I(a_0; a_1, a_2, a_3; a_4) = \int_{a_0}^{a_4} I(a_0; a_1, a_2; t) \frac{\mathrm{d}t}{t - a_3} \,, \tag{H.702}$$

and with the just determined weight-two integral, $I(a_0; a_1, a_2; t)$ we get

$$I(a_0; a_1, a_2, a_3; a_4) = G(a_3, a_2, a_1; a_4) - G(a_3, a_2, a_1; a_0) - G(a_1; a_0) \left[G(a_3, a_2; a_4) - G(a_3, a_2; a_0)\right]$$
$$+ \left[G(a_1; a_0) G(a_2; a_0) - G(a_2, a_1; a_0)\right] \left[G(a_3; a_4) - G(a_3; a_0)\right] \,. \tag{H.703}$$

H.53 Shuffle Regularisation

Regularise the MPL $G(z, z, a; z)$ wit $a \neq z$ using $G^{\text{reg}}(z, \ldots, z; z) = 0$.

Solution. Through the shuffle,

$$G(z, z, a; z) = G(z, z; z) G(a; z) - G(z, a, z; z) - G(a, z, z; z) \,, \tag{H.704}$$

the second term on the rhs is also divergent, since it has $a_1 = z$. We can shuffle it,

$$G(z, a, z; z) = G(z; z) G(a, z; z) - 2G(a, z, z; z) \,, \tag{H.705}$$

so

$$G(z, z, a; z) = G(z, z; z) G(a; z) - G(a, z, z; z) - G(z; z) G(a, z; z) + 2G(a, z, z; z)$$
$$= G(z, z; z) G(a; z) - G(z; z) G(a, z; z) + G(a, z, z; z) \,, \tag{H.706}$$

and then

$$G^{\text{reg}}(z, z, a; z) = G(a, z, z; z) \,. \tag{H.707}$$

H.54 Regularised Product

On the example $[G(z, a; z) G(b; z)]^{\text{reg}}$ show that the regularised product of two MPLs equals the product of the two regularised MPLs, i.e. the regularisation preserves the multiplication.

Solution. We have

$$[G(z,a;z)G(b;z)]^{\text{reg}} = \Big[G(z,a,b;z) + G(z,b,a;z) + G(b,z,a;z)\Big]^{\text{reg}}$$
$$= G^{\text{reg}}(z,a,b;z) + G^{\text{reg}}(z,b,a;z) + G(b,z,a;z)\,, \tag{H.708}$$

and we must regularise the first two terms (the third one is finite),

$$G(z,a,b;z) = G(z,z)G(a,b;z) - G(a,z,b;z) - G(a,b,z;z)\,, \tag{H.709}$$

so

$$G^{\text{reg}}(z,a,b;z) = -G(a,z,b;z) - G(a,b,z;z)\,, \tag{H.710}$$

likewise

$$G^{\text{reg}}(z,b,a;z) = -G(b,z,a;z) - G(b,a,z;z)\,, \tag{H.711}$$

thus

$$[G(z,a;z)G(b;z)]^{\text{reg}} = -G(a,z,b;z) - G(a,b,z;z) - G(b,a,z;z)$$
$$= -G(a,z;z)G(b;z)$$
$$= G^{\text{reg}}(z,a;z)G^{\text{reg}}(b;z)\,. \tag{H.712}$$

H.55 MPL Differentiation

Compute $\partial_y G(1,1+y;z)$.

Hint. $\partial_y G = \mu(\mathbb{1}\otimes\partial_y)\Delta_{n-1,1}(G)$ with $\mu(a\otimes b) = a\cdot b$ and the coproduct $\Delta(G(a_1,a_2;z))$ we computed in the lecture.

Solution. From $\Delta G(a_1,a_2;z)$ we computed in the lecture we have the element,

$$\Delta_{1,1}(G(a_1,a_2;z) = G(a_2;z)\otimes[G(a_1;z) - G(a_1;a_2)] + G(a_1;z)\otimes G(a_2;a_1)\,, \tag{H.713}$$

so

$$\Delta_{1,1}G(1,1+y;z) = G(1+y;z)\otimes[G(1;z) - G(1;1+y)] + G(1;z)\otimes G(1+y;1)\,, \tag{H.714}$$

and

$$\mu(\mathbb{1}\otimes\partial_y)\Delta_{1,1}G(1,1+y;z) = G(1+y;z)\partial_y[\cancel{G(1;z)} - G(1;1+y)] + G(1;z)\partial_y G(1+y;1)\,. \tag{H.715}$$

With

$$\partial_y G(1; 1+y) = \partial_y \log(1-(1+y)) = \frac{1}{y}\,, \tag{H.716}$$

$$\partial_y G(1+y; 1) = \partial_y \log\left(1 - \frac{1}{1+y}\right) = \partial_y \log\left(\frac{y}{1+y}\right) = \frac{1}{y(y+1)}\,. \tag{H.717}$$

we then get

$$\partial_y G(1, 1+y; z) = \frac{1}{y(1+y)} G(1; z) - \frac{1}{y} G(1+y; z)\,. \tag{H.718}$$

H.56 Coaction on the discontinuity

The discontinuity acts on the first entry of the coaction, $\Delta(\mathrm{Disc}\, G) = (\mathrm{Disc} \otimes \mathbb{1})\Delta(G)$. Verify that it hold on $G = \mathrm{Li}_2(z)$ and $G = \mathrm{Li}_3(z)$.

Hint. Use

$$\mathrm{Disc}(\mathrm{Li}_1(z)) = 2\pi i, \qquad \mathrm{Disc}(\mathrm{Li}_2(z)) = 2\pi i \log z, \qquad \mathrm{Disc}(\mathrm{Li}_3(z)) = 2\pi i \frac{\log^2 z}{2}\,. \tag{H.719}$$

Solution. For $G = \mathrm{Li}_2(z)$ the right-hand side is

$$\begin{aligned}(\mathrm{Disc} \otimes \mathbb{1})\, \Delta \mathrm{Li}_2(z) &= (\mathrm{Disc} \otimes \mathbb{1})\, (\mathrm{Li}_2(z) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{Li}_2(z) + \mathrm{Li}_1(z) \otimes \log z) \\ &= \mathrm{Disc}\, \mathrm{Li}_2(z) \otimes \mathbb{1} + \mathrm{Disc}\, \mathrm{Li}_1(z) \otimes \log z \\ &= 2\pi i \log z \otimes \mathbb{1} + 2\pi i \otimes \log z\,, \end{aligned} \tag{H.720}$$

and the left-hand side is

$$\begin{aligned}\Delta(\mathrm{Disc}\, \mathrm{Li}_2(z)) &= \Delta(2\pi i \log z) \\ &= \Delta(2\pi i)\Delta(\log z) \\ &= (2\pi i \otimes \mathbb{1})(\log z \otimes \mathbb{1} + \mathbb{1} \otimes \log z) \\ &= 2\pi i \log z \otimes \mathbb{1} + 2\pi i \otimes \log z\,. \end{aligned} \tag{H.721}$$

For the case $G = \mathrm{Li}_3(z)$ the right-hand side is

$$\begin{aligned}(\mathrm{Disc} \otimes \mathbb{1})\Delta \mathrm{Li}_3(z) &= (\mathrm{Disc} \otimes \mathbb{1}) \left(\mathbb{1} \otimes \mathrm{Li}_3(z) + \mathrm{Li}_3(z) \otimes \mathbb{1} + \mathrm{Li}_2(z) \otimes \log z + \mathrm{Li}_1(z) \otimes \frac{\log^2 z}{z}\right) \\ &= 2\pi i \frac{\log^2 z}{2} \otimes \mathbb{1} + 2\pi i \log z \otimes \log z + 2\pi i \otimes \frac{\log^2 z}{2}\,, \end{aligned} \tag{H.722}$$

and the left-hand side reads

$$
\begin{aligned}
\Delta(\mathrm{Disc}\,\mathrm{Li}_3(z)) &= \Delta\left(2\pi i\frac{\log^2 z}{2}\right) \\
&= \Delta(2\pi i)\Delta\left(\frac{\log^2 z}{2}\right) \\
&= (2\pi i\otimes \mathbb{1})\frac{1}{2}(\mathbb{1}\otimes\log z+\log z\otimes\mathbb{1})^2 \\
&= (2\pi i\otimes \mathbb{1})\frac{1}{2}(\mathbb{1}\otimes\log^2 z+\log^2 z\otimes\mathbb{1}+2\log z\otimes\log z) \\
&= 2\pi i\frac{\log^2 z}{2}\otimes\mathbb{1}+2\pi i\log z\otimes\log z+2\pi i\otimes\frac{\log^2 z}{2}\,.
\end{aligned}
\tag{H.723}
$$

H.57 A Dilogarithm Identity

Prove the identity,

$$
\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) - \log z\log(1-z) + \zeta_2\,. \tag{H.724}
$$

Solution. We have

$$
\Delta_{1,1}\mathrm{Li}_2(z) = \mathrm{Li}_1(z)\otimes\log z = -\log(1-z)\otimes\log z\,, \tag{H.725}
$$

$$
\Delta_{1,1}\mathrm{Li}_2(1-z) = -\log z\otimes\log(1-z)\,, \tag{H.726}
$$

$$
\Delta_{1,1}(\log z\log(1-z)) = \log z\otimes\log(1-z)+\log(1-z)\otimes\log z\,, \tag{H.727}
$$

so we can write

$$
\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)+\log z\log(1-z) = a\zeta_2\,. \tag{H.728}
$$

Setting $z=1$ we get $a=1$.

H.58 The Three-Mass Triangle

In sec. 2.6.11, it was mentioned that the three-mass triangle can be expressed in terms of a single-valued analogue of classical polylogarithms. In this exercise, we want to work towards this result, by solving the ϵ^0-coefficient in terms of iterated integrals. This will give us the opportunity to apply the techniques from sec. 2.6 to a non-trivial example.

The family of triangles with three external scales $p_i^0 \neq 0$ and $D = 4-2\epsilon$,

$$
T_{\nu_1,\nu_2,\nu_3}(p_1^2,p_2^2,p_3^2,\epsilon) = e^{\gamma_E}\int\frac{\mathrm{d}^D k}{i\pi^{\frac{D}{2}}}\frac{1}{(k^2)^{\nu_1}((k+p_1)^2)^{\nu}_2((k-p_2)^2)^{\nu_3}}\,, \tag{H.729}
$$

has the Feynman parametrisation,

$$
\begin{aligned}
&T_{\nu_1,\nu_2,\nu_3}(p_1^2,p_2^2,p_3^2,\epsilon)\\
&= C(\nu_i,\epsilon)\mathcal{I}_{\nu_1,\nu_2,\nu_3}(u,v,\epsilon)\\
&= C(\nu_i,\epsilon)(-p_3)^{D/2-\nu}\\
&\times \int\limits_{[0,\infty]^3} \left[\delta\left(1-\sum_{i\in S} x_i\right)\left(\sum_{j=1}^{3} x_j\right)^{\nu-D} (x_2x_3+x_1x_2u+x_1x_3v)^{D/2-\nu}\prod_{k=1}^{3} x_k^{\nu_k-1}\mathrm{d}x_k\right], \qquad \text{(H.730)}
\end{aligned}
$$

where $\nu=\nu_1+\nu_2+\nu_3$, S is a non-empty subset of $\{1,2,3\}$, $u=p_1^2/p_3^2$, $v=p_2^2/p_3^2$ and $C(\nu_i,\epsilon)$ is a prefactor depending only on the denominator powers and the dimension.

The relevant integral for the first Laurent coefficient and the choice $S=\{1\}$ is therefore

$$
I=\mathcal{I}^{(0)}_{1,1,1}(u,v)=\int\limits_{[0,\infty]^2}\frac{1}{(x_2+x_3+1)(x_2(u+x_3)+vx_3)}\mathrm{d}x_2\mathrm{d}x_3\,. \qquad \text{(H.731)}
$$

with $u,v>0$.

i) Perform the x_2 integration and write your result in terms of $G(a(u,v),x_3)$.

Hint. The variable change $y_2=\frac{x_2}{x_2+1}$ maps the domain of integration to the interval $[0,1]$.

Solution. To perform the integral we map with the suggested variable change to the unit interval and partial fraction in y_2, such that the integration in terms of logarithms is manifest,

$$
\begin{aligned}
I &= \int\limits_{[0,\infty]^2}\frac{1}{(x_2+x_3+1)(x_2(u+x_3)+vx_3)}\mathrm{d}x_2\mathrm{d}x_3\\
&= \int\limits_{[0,\infty]}\int\limits_0^1\frac{1}{(x_3(y_2-1)-1)(vx_3(y_2-1)-y_2(u+x_3))}\mathrm{d}y_2\mathrm{d}x_3\\
&= \int\limits_{[0,\infty]} -\frac{1}{x_3(u-v+1)+u+x_3^2}\int\limits_0^1\Big(\frac{(v-1)x_3-u}{uy_2+x_3\,(v-(v-1)y_2)}+\frac{x_3}{x_3\,(y_2-1)-1}\Big)\mathrm{d}y_2\mathrm{d}x_3\\
&= \int\limits_{[0,\infty]} -\frac{1}{x_3(u-v+1)+u+x_3^2}\left(-\log\left(u+x_3\right)+\log v+\log x_3-\log\left(x_3+1\right)\right)\mathrm{d}x_3\\
&= -\int_{[0,\infty]}\frac{-G(-u;x_3)+G(0;v)-G(-1;x_3)+G(0;x_3)-G(0;u)}{x_3(u-v+1)+u+x_3^2}\mathrm{d}x_3\,, \qquad \text{(H.732)}
\end{aligned}
$$

where we used the rescaling invariance $G(\vec{a};x)=G(y\vec{a};yx)$ for

$$
\log(u+x_3)=\log u+G(-1;\frac{x_3}{u})=\log u+G(-u;x_3)\,. \qquad \text{(H.733)}
$$

ii) After integration of x_2 we are left with the integral

$$I = -\int_{[0,\infty]} \frac{-G(-u;x_3) + G(0;v) - G(-1;x_3) + G(0;x_3) - G(0,u)}{x_3(u-v+1) + u + x_3^2} \mathrm{d}x_3 \,. \tag{H.734}$$

In order to perform this integral we want to find a primitive completely expressed in terms of $G(\vec{w}, x_3)$ which we need to evaluate in the two limits $x_3 \to 0$ and $x_3 \to \infty$. The limit $x \to 0$ is trivial, since $\lim_{x\to 0} G(\vec{w};x) = 0$ if $\vec{w} \neq \vec{0}$. The challenging limit is $x_3 \to \infty$ and in order to take it we need to find a representation $G(\vec{w};x) = \sum_i G(\vec{w}_i, 1/x)$ such that the limit on the r.h.s is $\bar{x} = 1/x \to 0$.

In the following we want to get an idea on the inversion of the argument of a MPL by deriving the relation,

$$G(-z,0;x_3) = -G\left(-\frac{1}{z},0;\frac{1}{x_3}\right) + G\left(0,0;\frac{1}{x_3}\right) - G(0,0;z) - \frac{\pi^2}{6} \,. \tag{H.735}$$

For that, we study the MPL,

$$G\left(a,0;\frac{1}{x}\right) , \tag{H.736}$$

and reconstruct from the symbol tensor the inversion relation.

- Compute the co-product $\Delta_{1,1} G\left(a,0;\frac{1}{x}\right)$ and expand the logarithms such that they are all of the form $\log(c_1 + c_2 x)$.

  Solution. From eq. (2.129), we know that

$$\Delta_{1,1}(G(a_1,a_2;z)) = G(a_1;z) \otimes G(a_2;a_1) + G(a_2;z) \otimes G(a_1;z) - G(a_2;z) \otimes G(a_1;a_2) \,. \tag{H.737}$$

  Since in our case $a_2 = 0$, the last term will vanish and we have

$$\begin{aligned} \Delta_{1,1} G\left(a,0;\frac{1}{x}\right) &= G\left(0;\frac{1}{x}\right) \otimes G\left(a;\frac{1}{x}\right) + G\left(a;\frac{1}{x}\right) \otimes G(0;a) \\ &= \log\left(\frac{1}{x}\right) \otimes \log\left(1 - \frac{1}{ax}\right) + \log\left(1 - \frac{1}{ax}\right) \otimes \log a \\ &= -\log x \otimes \log(ax-1) + \log(ax-1) \otimes \log a - \log a \otimes \log a \\ &\quad + \log x \otimes \log x \,. \end{aligned} \tag{H.738}$$

- Read off the symbol tensor,

$$T = \mathcal{S}\left(G\left(a,0;\frac{1}{x}\right)\right) = \Delta_{1,1} G\left(a,0;\frac{1}{x}\right) \mod i\pi \,. \tag{H.739}$$

Hint. Notice the $\mod i\pi$. It means e.g. that

$$\log(ax-1)\otimes\log(1-ax)\to(1-ax)\otimes(1-ax)\,, \tag{H.740}$$

$$\log(1-ax)\otimes\log(1-ax)\to(1-ax)\otimes(1-ax)\,, \tag{H.741}$$

and so on.

Solution. We can read off the symbol directly from the co-product,

$$T=\mathcal{S}\left(G\left(a,0,\frac{1}{x}\right)\right)=-x\otimes(1-ax)+(1-ax)\otimes a-a\otimes a+x\otimes x\,, \tag{H.742}$$

where we used the mod $i\pi$ to write the entries as $(1-ax)$.

- We now associate to a given symbol s a MPL with the map,

$$\Phi_{x_i}(s)=\begin{cases}G\Big(-\frac{b_1}{a_1},\dots,-\frac{b_w}{a_w};x_i\Big), & \text{if } s=(a_wx_i+b_w)\otimes\ldots\otimes(a_1x_i+b_1),\ \forall\ a_i\neq 0\\ 0, & \text{else}\end{cases}. \tag{H.743}$$

Apply the map Φ_x to the previously computed symbol $T(x,s)$ and compute the symbol of the result $T_x=\mathcal{S}(\Phi_x(T))$.

Solution. Applying the map Φ_x to T gives

$$\begin{aligned}\Phi_x(T)&=-\Phi_x(x\otimes(1-ax))+\Phi_x((1-ax)\otimes a)-\Phi_x(a\otimes a)+\Phi_x(x\otimes x)\\ &=-G\left(\frac{1}{a},0;x\right)+0+0+G(0,0;x)\,,\end{aligned} \tag{H.744}$$

and the symbol T_x is

$$T_x=\mathcal{S}(\Phi_x(T))=-x\otimes(1-ax)+(1-ax)\otimes a+x\otimes x\,, \tag{H.745}$$

which is obtained in a completely equivalent way to the previous computation.

- What you saw in the previous task: $\Phi_{x_i}(T)$ will give a MPL $G(\star,x_i)$ with a symbol T_{x_i} which has all the terms which have x_i in all entries as T.
Compute $T_2=T-T_x$ and apply $\Phi_a(T_2)$. Compare $\tilde{G}=\Phi_x(T)+\Phi_a(T_2)$ with (H.735). How would you reconstruct the missing constant?
Hint. We have $\mathcal{S}(G(0,0,-a))=\mathcal{S}(G(0,0,a))=a\otimes a$. Take the one which for $a<0$ is real. You can furthermore use $G(-1,0;1)=-\frac{\pi}{12}$.
Solution. We see

$$T_2=T-T_x=-a\otimes a\,, \tag{H.746}$$

so $G(0,0;-a)$ is a possible choice for the polylogarithm with the symbol $a\otimes a$ which is real for $a < 0$. This is of course a shortcut. If $G(0, 0, a)$ is chosen one has to use the discontinuity to reconstruct $-G(0, 0; a) + i\pi G(0; a)$. So the final answer,

$$\tilde{G} = \Phi_x(T) + \Phi_a(T_2) = -G\left(\frac{1}{a}, 0; x\right) + G(0, 0; x) - G(0, 0; -a)\,, \tag{H.747}$$

which is real for $a < 0$ and $x > 0$ and if we compare with (H.735) we see that $G(a, 0, \frac{1}{x}) - \tilde{G}$ is a constant (of weight two. So it has to be proportional to π^2). To find the constant we can use the point $x = 1, a = -1$ for which

$$G(-1, 0; 1) - [\, -G(-1, 0; 1) + \cancelto{0}{G(0, 0; 1)} - \cancelto{0}{G(0, 0; 1)}] = 2G(-1, 0; 1) = -\frac{\pi^2}{6}\,. \tag{H.748}$$

In practice, finding a special point for a large multi-variable expression without investigating the expression by eye is difficult. But one can reconstruct the constants by comparing with high precision (usually 100 Digits) at a point where both expressions are real. Since the space of allowed constants is predetermined this reconstruction is bound to work. E.g. at the point $z = 1/3$ and $x = 1/7$ one would find

$$G(-\frac{1}{3}, 0, 7) = -1.08379541121643925253009968992008932984\,, \tag{H.749}$$

$$\tilde{G}|_{z=\frac{1}{3}, x=\frac{1}{7}} = 0.56113865563178718394231547672593585937\,. \tag{H.750}$$

and since the only allowed constant is ζ_2, the fit is trivial.

The evaluation of polylogarithms can be performed with a code called GiNaC [5]. What we did to find the transformation is often referred to as finding a "fibration basis". It is usually needed not only for computing limits, but for rewriting polylogarithms with letters $a_i(y)$ depending on the a integration variable y in a form, where integration variable appears only in the argument of all MPLs and not in the letters anymore. The way we did it here is a (easy) version of the method presented in the computation of real radiation contributions for Higgs production [62]. A different way of finding a fibration basis which does not involve the fitting of rational constants is described in the thesis of Erik Panzer[6] and implemented in his public program HyperInt. In practice, HyperInt is the best tool for finding a fibration basis.

iii) Perform the remaining integration,

$$I = -\int_{[0,\infty]} \frac{-G(-u; x_3) + G(0; v) - G(-1; x_3) + G(0; x_3) - G(0, u)}{x_3(u - v + 1) + u + x_3^2} \mathrm{d}x_3\,, \tag{H.751}$$

without mapping to the unit interval but directly in terms of MPLs. Use the variable change $u = z\bar{z}$ and $v = (1 - \bar{z})(1 - z)$ for the denominator. In particular verify that all the GPLs of

[5] https://ginac.de/

[6] See sec. 3.6.2 of ref. [63].

the form $\lim_{x\to 0} G(\vec{0}, x)$ cancel explicitly if the sum over all terms is considered. Why does that need to be the case?

Hint. The inversion identities are

$$G(-z;x_3) = G(-\frac{1}{z};\frac{1}{x_3}) - G(0;\frac{1}{x_3}) - G(0;z)\,, \tag{H.752}$$

$$G(-\bar{z};x_3) = G(-\frac{1}{\bar{z}};\frac{1}{x_3}) - G(0;\frac{1}{x_3}) - G(0;\bar{z})\,, \tag{H.753}$$

$$G(-z,0;x_3) = -G(-\frac{1}{z},0;\frac{1}{x_3}) + G(0,0;\frac{1}{x_3}) - G(0,0;z) - \frac{\pi^2}{6}\,, \tag{H.754}$$

$$G(-\bar{z},0;x_3) = -G(-\frac{1}{\bar{z}},0;\frac{1}{x_3}) + G(0,0;\frac{1}{x_3}) - G(0,0;\bar{z}) - \frac{\pi^2}{6}\,, \tag{H.755}$$

$$\begin{aligned} G(-z,-1;x_3) = {}& G(-\frac{1}{z},-1;\frac{1}{x_3}) - G(-\frac{1}{z},0;\frac{1}{x_3}) - G(0,-1;\frac{1}{x_3}) + G(0,0;\frac{1}{x_3}) - G(1,0;z) \\ & + \frac{\pi^2}{6}\,, \end{aligned} \tag{H.756}$$

$$\begin{aligned} G(-\bar{z},-1;x_3) = {}& G(-\frac{1}{\bar{z}},-1;\frac{1}{x_3}) - G(-\frac{1}{\bar{z}},0;\frac{1}{x_3}) - G(0,-1;\frac{1}{x_3}) + G(0,0;\frac{1}{x_3}) - G(1,0;\bar{z}) \\ & + \frac{\pi^2}{6}\,, \end{aligned} \tag{H.757}$$

$$\begin{aligned} G(-\bar{z},-u;x_3) = {}& -G(0;u)G(-\frac{1}{\bar{z}};\frac{1}{x_3}) + G(-\frac{1}{\bar{z}},-\frac{1}{u};\frac{1}{x_3}) + G(0;u)G(0;\frac{1}{x_3}) - G(0,-\frac{1}{u};\frac{1}{x_3}) \\ & + G(0;u)G(0;\bar{z}) - G(0;\bar{z})G(\bar{z};u) + G(\bar{z},0;u) - G(-\frac{1}{\bar{z}},0;\frac{1}{x_3}) + G(0,0;\frac{1}{x_3}) \\ & - G(0,0;\bar{z}) - \frac{\pi^2}{6}\,, \end{aligned} \tag{H.758}$$

$$\begin{aligned} G(-z,-u;x_3) = {}& -G(0;u)G(-\frac{1}{z};\frac{1}{x_3}) + G(-\frac{1}{z},-\frac{1}{u};\frac{1}{x_3}) + G(0;u)G(0;\frac{1}{x_3}) - G(0,-\frac{1}{u};\frac{1}{x_3}) \\ & + G(0;u)G(0;z) - G(0;z)G(z;u) + G(z,0;u) - G(-\frac{1}{z},0;\frac{1}{x_3}) + G(0,0;\frac{1}{x_3}) \\ & - G(0,0;z) - \frac{\pi^2}{6}\,. \end{aligned} \tag{H.759}$$

Notice that you are only interested in the case $x_3 \to \infty$ where many of the terms drop out since $\lim_{x=0} G(\vec{a}, x) = 0$ for $\vec{a} \neq \vec{0}$.

Solution. Performing the variable change and applying partial fractioning we obtain

$$\begin{aligned} I &= \int_{[0,\infty]} \frac{G(-u;x_3) - G(0;v) + G(-1;x_3) - G(0;x_3) + G(0,u)}{(x_3+z)(x_3+\bar{z})} \mathrm{d}x \\ &= \int_{[0,\infty]} \Big[\frac{-G(-u;x_3) + G(0;v) - G(-1;x_3) + G(0;x_3) - \log(u)}{(x_3+z)(z-\bar{z})} \\ &\qquad + \frac{G(-u;x_3) - G(0;v) + G(-1;x_3) - G(0;x_3) + G(0,u)}{(x_3+\bar{z})(z-\bar{z})}\Big] \mathrm{d}x\,. \end{aligned} \tag{H.760}$$

So we will get the new letter $-z$ for every $G(\star;x_3)$ in the first term and $-\bar{z}$ for the second one. We find

$$\tilde{I}(x_3) = -\frac{(G(-z,-u;x_3) - G(-\bar{z},-u;x_3))}{z-\bar{z}} - \frac{G(0;u)(G(-z;x_3) - G(-\bar{z};x_3))}{z-\bar{z}} + \frac{G(0;v)(G(-z;x_3) - G(-\bar{z};x_3))}{z-\bar{z}} - \frac{(G(-z,-1;x_3) - G(-\bar{z},-1;x_3))}{z-\bar{z}} + \frac{(G(-z,0;x_3) - G(-\bar{z},0;x_3))}{z-\bar{z}}, \tag{H.761}$$

which needs to be evaluated at $x_3 \to \infty$ ($x_3 \to 0$ gives 0 as of the hint). The leading behaviour of the MPLs as $x_3 \to \infty$ can be obtained from the inversion relations by using the limit $\bar{x} \to 0$ for $\bar{x} = \frac{1}{x_3}$,

$$G(-z;\frac{1}{\bar{x}}) = -G(0;\bar{x}) - G(0;z), \tag{H.762}$$

$$G(-\bar{z};\frac{1}{\bar{x}}) = -G(0;\bar{x}) - G(0;\bar{z}), \tag{H.763}$$

$$G(-z,0;\frac{1}{\bar{x}}) = \frac{1}{6}(3G(0;\bar{x})^2 - 6G(0,0;z) - \pi^2), \tag{H.764}$$

$$G(-\bar{z},0;\frac{1}{\bar{x}}) = \frac{1}{6}(3G(0;\bar{x})^2 - 6G(0,0;\bar{z}) - \pi^2), \tag{H.765}$$

$$G(-z,-1;\frac{1}{\bar{x}}) = \frac{1}{6}(3G(0;\bar{x})^2 - 6G(1,0;z) + \pi^2), \tag{H.766}$$

$$G(-\bar{z},-1;\frac{1}{\bar{x}}) = \frac{1}{6}(3G(0;\bar{x})^2 - 6G(1,0;\bar{z}) + \pi^2), \tag{H.767}$$

$$G(-z,-u;\frac{1}{\bar{x}}) = G(0;u)(G(0;\bar{x}) + G(0;z)) - G(0;z)G(z;u) + G(z,0;u) + \frac{1}{2}G(0;\bar{x})^2 - G(0,0;z) - \frac{\pi^2}{6}, \tag{H.768}$$

$$G(-\bar{z},-u;\frac{1}{\bar{x}}) = G(0;u)(G(0;\bar{x}) + G(0;\bar{z})) - G(0;\bar{z})G(\bar{z};u) + G(\bar{z},0;u) + \frac{1}{2}G(0;\bar{x})^2 - G(0,0;\bar{z}) - \frac{\pi^2}{6}. \tag{H.769}$$

In particular, notice that the in eq. (H.761) the sign between $G(\bar{z},\star)$ and $G(z,\star)$ is such that the divergent $G(\vec{0},x_3)$ cancel pairwise, and one finds

$$I = \frac{1}{z-\bar{z}}\Big[\log z\ (G(z;u) + \log(1-z)) - \log\bar{z}\ (G(\bar{z};u) + \log(1-\bar{z})) - G(z,0;u) + G(\bar{z},0;u) + \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \log v\ (\log\bar{z} - \log z)\Big]. \tag{H.770}$$

If you compute Feynman integrals via Feynman parameters you should always find the cancellation of the end-point singularities. If it does not happen it means you expanded the integrand in the dimensional regulator even though the integral is not finite, which of course would be wrong. For most integrals you will find that the end-point singularities exist and you are not allowed to expand in ϵ before performing the integration. Since these endpoint singularities are a consequence of IR-singularities, for on-shell kinematics and massless theories you will have them in a naive basis-choice of MIs almost all the time. However, they can often be circumvented by finding what is sometimes referred to as "quasi finite basis". This is based on the work of Erik Panzer, however beyond the scope of the class.

We can simplify our expression even more by inserting $u = z\bar{z}$ and $v = (1-\bar{z})(1-z)$ and computing a fibration with respect to the variables z and $\bar{z}$. This is similar to what you have done for the inversion of the argument $G(\star;\frac{1}{\bar{x}}) \to \sum_i G(\star_i,\bar{x})$ by applying the maps Φ_z and $\Phi_{\bar{z}}$ and by taking care of factorized terms of the form $G(\star,\bar{z})G(\tilde{\star},z)$. Since we already did a similar computation there is no need to do it by hand again. One can easily do it with either the program HyperInt[7] or PolyLogTools[8]. Alternatively it can be obtained with the decomposition of $G(a,b,\bar{x})$ in terms of Li_2 where one has to be a bit careful with the limits. What you will find is

$$\begin{aligned} I &= \frac{-G(0;z)(G(1;z)+G(1;\bar{z}))+G(0;\bar{z})(G(1;z)+G(1;\bar{z}))+2(G(1,0;z)-G(1,0;\bar{z}))}{z-\bar{z}} \\ &= \frac{2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar{z})+(\log(1-z)-\log(1-\bar{z}))(\log z+\log\bar{z})}{z-\bar{z}}\,. \end{aligned} \tag{H.771}$$

In particular notice the following: for complex variables z and $\bar{z}=z^*$ the expression $(\log z+\log\bar{z}) = \log|z|$, $\log(1-z)-\log(1-\bar{z}) = 2i\Im(\log(1-z)) = -2i\Im(\mathrm{Li}_1(z))$ and $\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z}) = 2i\Im(\mathrm{Li}_2(z))$ are all single valued and we have

$$\begin{aligned} I &= \frac{2i}{z-\bar{z}}\Im\left(-\log(|z|)\mathrm{Li}_1(z)+\mathrm{Li}_2(z)\right) \\ &= \frac{2i}{z-\bar{z}}D_2(z)\,, \end{aligned} \tag{H.772}$$

where the function D_2 is the single-valued analogue of the classical polylogarithm defined by Zagier [55], which was alluded too in sec. 2.6.11.

H.59 $G(a,b;z)$ in terms of $\mathrm{Li}_2(z)$

Prove that for generic a, b, x ($a \neq b \neq x \neq a$ and none of them zero):

$$G(a,b;x) = \mathrm{Li}_2\left(\frac{b-x}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \log\left(1-\frac{x}{b}\right)\log\left(\frac{x-a}{b-a}\right)\,. \tag{H.773}$$

using either $\Delta_{1,1}G$, or Goncharov's formula (2.167) for the symbol,

$$\mathcal{S}(I(a_0;a_1,\ldots,a_n;a_{n+1})) = \sum_{i=1}^{n}\mathcal{S}(I(a_0;a_1,\ldots,\not{a_i},\ldots,a_n;a_{n+1}))\otimes\log\left(\frac{a_{i+1}-a_i}{a_{i-1}-a_i}\right)\,.$$

[7]https://arxiv.org/abs/1403.3385
[8]https://arxiv.org/pdf/1904.07279.pdf

Solution. We compute the symbol,

$$\begin{aligned}\mathcal{S}\left(I(0;a_1,a_2;a_3)\right) &= \mathcal{S}\left(I(0;a_2;a_3)\right)\otimes\log\left(\frac{a_2-a_1}{-a_1}\right)+\mathcal{S}\left(I(0;a_1;a_3)\right)\otimes\log\frac{a_3-a_2}{a_1-a_2}\\ &=\log\frac{a_3-a_2}{-a_2}\otimes\log\frac{a_2-a_1}{-a_1}+\log\frac{a_3-a_1}{-a_1}\otimes\log\frac{a_3-a_2}{a_1-a_2}\,,\end{aligned}\tag{H.774}$$

and set $a_1=b$, $a_2=a$ and $a_3=x$ (remember the conventions on the reversed letters of the I-functions),

$$S(G(a,b;x))=\underbrace{\frac{a-x}{a}\otimes\frac{b-a}{b}}_{(i)}+\underbrace{\frac{b-x}{b}\otimes\frac{a-x}{a-b}}_{(ii)}\tag{H.775}$$

We compute the symbols of the right-hand side,

$$\begin{aligned}\Delta_{1,1}\mathrm{Li}_2\left(\frac{b-x}{b-a}\right)&=-\log\left(1-\frac{b-x}{b-a}\right)\otimes\log\frac{b-x}{b-a}\\ &=-\log\frac{a-x}{a-b}\otimes\log\left(-\frac{b-x}{a-b}\right)\,,\end{aligned}\tag{H.776}$$

$$\begin{aligned}\Delta_{1,1}\mathrm{Li}_2\left(\frac{b}{b-a}\right)&=-\log\left(1-\frac{b}{b-a}\right)\otimes\log\frac{b}{b-a}\\ &=-\log\left(-\frac{a}{b-a}\right)\otimes\log\frac{b}{b-a}\,,\end{aligned}\tag{H.777}$$

$$\begin{aligned}\Delta_{1,1}\left(\log\left(1-\frac{x}{b}\right)\log\frac{x-a}{b-a}\right)&=\log\frac{x-a}{b-a}\otimes\log\left(1-\frac{x}{b}\right)+\log\left(1-\frac{x}{b}\right)\otimes\log\frac{x-a}{b-a}\\ &=\log\frac{a-x}{a-b}\otimes\log\frac{b-x}{b}+\log\frac{b-x}{b}\otimes\log\frac{a-x}{a-b}\,,\end{aligned}\tag{H.778}$$

and we see, that (ii) from (H.775) is part of the symbol of $\log\left(1-\frac{x}{b}\right)\log\frac{x-a}{b-a}$. So we just have to investigate the remaining terms of the right-hand side indeed give (i):

$$\begin{aligned}(i)&\overset{?}{=}-\frac{a-x}{a-b}\otimes\left(-\frac{b-x}{a-b}\right)+\left(-\frac{a}{b-a}\right)\otimes\frac{b}{b-a}+\frac{a-x}{a-b}\otimes\frac{b-x}{b}\\ &=-\frac{a}{a-b}\left(\frac{a-x}{a}\right)\otimes\frac{b}{b-a}\left(\frac{b-x}{b}\right)+\frac{a}{a-b}\otimes\frac{b}{b-a}\\ &\quad+\frac{a}{a-b}\left(\frac{a-x}{a}\right)\otimes\frac{b-x}{b}\\ &=-\left(\cancel{\frac{a}{a-b}\left(\frac{a-x}{a}\right)\otimes\left(\frac{b-x}{b}\right)}+\frac{a}{a-b}\left(\frac{a-x}{a}\right)\otimes\frac{b}{b-a}\right)\\ &\quad+\frac{a}{a-b}\otimes\frac{b}{b-a}\cancel{+\frac{a}{a-b}\left(\frac{a-x}{a}\right)\otimes\left(\frac{b-x}{b}\right)}\\ &=-\left(\cancel{\frac{a}{a-b}\otimes\frac{b}{b-a}}+\frac{a-x}{a}\otimes\frac{b}{b-a}\right)+\cancel{\left(\frac{a}{a-b}\right)\otimes\left(\frac{b}{b-a}\right)}\\ &=\frac{a-x}{a}\otimes\frac{b-a}{b}=(i)\,,\end{aligned}\tag{H.779}$$

and we see that left-hand and right-hand sides indeed agree on the symbol level. The agreement on the symbol level however, is only a necessary condition and we still have to verify that we are not missing a constant term $a\zeta_2$ which would be in the kernel of the symbol map. To check that we can set $a = 0$ and $b = 1$ and evaluate

$$\begin{aligned} G(0,1;x) &= \mathrm{Li}_2\left(1-x\right) - \mathrm{Li}_2(1) + \log\left(1-x\right)\log x + a\zeta_2 \\ &= \mathrm{Li}_2\left(1-x\right) - \zeta_2 + \log\left(1-x\right)\log x + a\zeta_2\,, \end{aligned} \tag{H.780}$$

with

$$\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) - \log x\log(1-x) + \zeta_2\,, \tag{H.781}$$

we get

$$G(0,1;x) = -\mathrm{Li}_2(x) + a\zeta_2\,, \tag{H.782}$$

so $a = 0$ and the proof is complete.

H.60 Iterated Integration: Assigning Functions to Symbols

Consider the integral,

$$I = \int_{[0,1]^2} G\left(\frac{(x-1)}{1-y}, \frac{1}{y}; x\right) \frac{\mathrm{d}x}{x+1}\frac{\mathrm{d}y}{y+1}\,. \tag{H.783}$$

We can not yet integrate it in terms of MPLs since it is not of the form $G(a_1(x), a_2(x); y)$ or $G(a_1(y), a_2(y); x)$.

In order to bring it in one of the forms which can be integrated, we want to apply the approach outlined in the exercise class.

i) Compute the symbol of $G\left(\frac{(x-1)}{1-y}, \frac{1}{y}; x\right)$.

Solution. The symbol of a weight-two MPL was already computed on the last exercise sheet and the computation here is completely analogous. We have

$$\begin{aligned}\Delta_{1,1}G\left(\frac{(x-1)}{1-y},\frac{1}{y};x\right)=&G\left(\frac{x-1}{1-y};x\right)\otimes G\left(\frac{1}{y};\frac{x-1}{1-y}\right)+G\left(\frac{1}{y};x\right)\otimes G\left(\frac{x-1}{1-y};x\right)\\&-G\left(\frac{1}{y};x\right)\otimes G\left(\frac{x-1}{1-y};\frac{1}{y}\right)\\=&\log\left(1-\frac{x(1-y)}{x-1}\right)\otimes\log\left(1-\frac{(x-1)y}{1-y}\right)\\&+\log(1-xy)\otimes\log\left(1-\frac{x(1-y)}{x-1}\right)-\log(1-xy)\otimes\log\left(1-\frac{1-y}{(x-1)y}\right)\\=&\log(1-xy)\otimes\log\left(\frac{xy-1}{x-1}\right)-\log(1-xy)\otimes\log\left(\frac{xy-1}{(x-1)y}\right)\\&+\log\left(\frac{xy-1}{x-1}\right)\otimes\log\left(\frac{xy-1}{y-1}\right),\end{aligned}\tag{H.784}$$

so

$$\begin{aligned}\mathcal{S}\left(G\left(\frac{(x-1)}{1-y},\frac{1}{y};x\right)\right)=&(1-x)\otimes(1-y)-(1-x)\otimes(1-xy)-(1-xy)\otimes(1-y)\\&+(1-xy)\otimes y+(1-xy)\otimes(1-xy).\end{aligned}\tag{H.785}$$

ii) In the exercise class, we discussed how to assign a “canonical” function to the symbol, such that for a chosen ordering on the variables $x_1\prec x_2\prec\ldots\prec x_n$ the argument of the function is x_i and all the letters depend on $x_j\succ x_i$ only. The maps which assign such a function,

1). are linear in x_i, for all w entries of the symbol,

$$\Phi_{x_i}(s)=\begin{cases}G\Big(-\frac{b_1}{a_1},\ldots,-\frac{b_w}{a_w};x_i\Big), & \text{if } s=(a_wx_i+b_w)\otimes\ldots\otimes(a_1x_i+b_1),\ \forall\ a_i\neq 0\\ 0, & \text{else}\end{cases}\tag{H.786}$$

2). are linear in x_i for the last $w-k$ entries of the symbol, and depend only on $x_j\succ x_i$ for the first k entries,

$$\begin{aligned}&\varPhi_{x_j,x_i}\Big(b_1(x_j)\otimes\ldots\otimes b_k(x_j)\otimes(a_{k+1}x_i+b_{k+1})\otimes\ldots\otimes(a_wx_i+b_w)\Big)\\&=\Phi_{x_j}\Big(b_1(x_j)\otimes\ldots\otimes b_k(x_j)\Big)\times\Phi_{x_i}\Big((a_{k+1}x_i+b_{k+1})\otimes\ldots\otimes(a_wx_i+b_w)\Big).\end{aligned}\tag{H.787}$$

In the previous exercise you computed

$$\begin{aligned}T=\mathcal{S}\left(G\left(\frac{(x-1)}{1-y},\frac{1}{y};x\right)\right)=&(1-x)\otimes(1-y)-(1-x)\otimes(1-xy)\\&-(1-xy)\otimes(1-y)+(1-xy)\otimes y\\&+(1-xy)\otimes(1-xy).\end{aligned}\tag{H.788}$$

Consider the ordering $x \prec y$ and build iteratively the "canonical" function $\mathcal{G}(x)$ which has the same symbol T:

(a) Compute $\mathcal{G}_1(x) = \Phi_x(T)$ and $T_1 = T - \mathcal{S}(\mathcal{G}_1(x))$.

Solution. The symbol

$$\begin{aligned} T =&(1-x)\otimes(1-y) - \underbrace{(1-x)\otimes(1-xy)}_{(i)} \\ &-(1-xy)\otimes(1-y) + (1-xy)\otimes y \\ &+\underbrace{(1-xy)\otimes(1-xy)}_{(ii)} \,. \end{aligned} \tag{H.789}$$

has the two indicated terms which are not in the kernel of Φ_x. We have

$$(i): \qquad \Phi_x\Big((1-x)\otimes(1-xy)\Big) = G\left(\frac{1}{y},1;x\right)\,, \tag{H.790}$$

$$(ii): \qquad \Phi_x\Big((1-xy)\otimes(1-xy)\Big) = G\left(\frac{1}{y};\frac{1}{y},x\right)\,, \tag{H.791}$$

so

$$\mathcal{G}_1(x) = -G\left(\frac{1}{y},1;x\right) + G\left(\frac{1}{y};\frac{1}{y},x\right)\,, \tag{H.792}$$

with

$$\begin{aligned} \mathcal{S}\left(G\left(\frac{1}{y},1;x\right)\right) =& -((1-x)\otimes(1-y)) + (1-x)\otimes(1-xy) + (1-xy)\otimes(1-y) \\ &-(1-xy)\otimes y\,, \end{aligned} \tag{H.793}$$

$$\mathcal{S}\left(G\left(\frac{1}{y},\frac{1}{y};x\right)\right) =(1-xy)\otimes(1-xy)\,, \tag{H.794}$$

we have

$$\begin{aligned} \mathcal{S}(\mathcal{G}_1) =&(1-x)\otimes(1-y) - (1-x)\otimes(1-xy) \\ &-(1-xy)\otimes(1-y) + (1-xy)\otimes y + (1-xy)\otimes(1-xy)\,, \end{aligned} \tag{H.795}$$

which is exactly T. So $T_1 = T - \mathcal{S}(\mathcal{G}_1) = 0$ and we won't need further iterations.

(b) If necessary, proceed with the iteration and compute $\mathcal{G}_2(x) = \mathit{\Phi}_{y,x}(T_1)$ and $T_2 = T_1 - \mathcal{S}(\mathcal{G}_2(x))$.

Solution. Not needed for $x \prec y$.

(c) If necessary, proceed with the iteration and compute $\mathcal{G}_3(y) = \Phi_y(T_2)$ and $T_3 = T_2 - \mathcal{S}(\mathcal{G}_3(y))$.

Solution. Not needed for $x \prec y$.

(d) Can you conclude

$$G\left(\frac{(x-1)}{1-y},\frac{1}{y};x\right)=\mathcal{G}_1+\mathcal{G}_2+\mathcal{G}_3? \tag{H.796}$$

Solution. It is important to always keep in mind, that such a conclusion is not allowed, since the construction is always modulo terms which are in the kernel of the symbol map. However, we can verify that the relation is exact by e.g. looking at the case $x \to 0$, where both sides go to 0.

iii) Perform the same computation as before for the ordering $y \prec x$. What do you see?

Solution. The symbol

$$\begin{aligned} T =&(1-x)\otimes(1-y)-(1-x)\otimes(1-xy)\\ &-\underbrace{(1-xy)\otimes(1-y)}_{(i)}+\underbrace{(1-xy)\otimes y}_{(ii)}\\ &+\underbrace{(1-xy)\otimes(1-xy)}_{(iii)}\,. \end{aligned} \tag{H.797}$$

has the three indicated terms which are not in the kernel of Φ_y.

$$(i): \qquad \Phi_y\Big((1-xy)\otimes(1-y)\Big)=G(1,\frac{1}{x};y)\,, \tag{H.798}$$

$$(ii): \qquad \Phi_y\Big((1-xy)\otimes y\Big)=G(0,\frac{1}{x};y)\,, \tag{H.799}$$

$$(iii): \qquad \Phi_y\Big((1-xy)\otimes(1-xy)\Big)=G(\frac{1}{x},\frac{1}{x};y)\,, \tag{H.800}$$

so

$$\mathcal{G}_1(y)=-G(1,\frac{1}{x};y)+G(0,\frac{1}{x};y)+G(\frac{1}{x},\frac{1}{x};y)\,. \tag{H.801}$$

With the symbols

$$\mathcal{S}\left(G(1,\frac{1}{x};y)\right)=(1-y)\otimes(1-x)-(1-xy)\otimes(1-x)+(1-xy)\otimes x+(1-xy)\otimes(1-y)\,, \tag{H.802}$$

$$\mathcal{S}\left(G(0,\frac{1}{x};y)\right)=(1-xy)\otimes x+(1-xy)\otimes y\,, \tag{H.803}$$

$$\mathcal{S}\left(G(\frac{1}{x},\frac{1}{x};y)\right)=(1-xy)\otimes(1-xy)\,, \tag{H.804}$$

we get

$$\begin{aligned} \mathcal{S}(\mathcal{G}_1)=&-((1-y)\otimes(1-x))+(1-xy)\otimes(1-x)-(1-xy)\otimes(1-y)+(1-xy)\otimes y\\ &+(1-xy)\otimes(1-xy)\,, \end{aligned} \tag{H.805}$$

and

$$
\begin{aligned}
T_1 =& T - \mathcal{S}(\mathcal{G}_1) \\
=& \underbrace{(1-x) \otimes (1-y)}_{(iv)} - \underbrace{(1-x) \otimes (1-xy)}_{(v)} + (1-y) \otimes (1-x) - (1-xy) \otimes (1-x) \,. \quad \text{(H.806)}
\end{aligned}
$$

Since the symbol $T_1 \neq 0$, we have to perform the next step in the iteration: applying the map $\Phi_{x,y}$. The two indicated terms contribute and we have

$$
(iv): \qquad \Phi_{x,y}((1-x) \otimes (1-y)) = G(1;x)G(1;y)\,, \tag{H.807}
$$

$$
(v): \qquad \Phi_{x,y}((1-x) \otimes (1-xy)) = G(1;x)G(\frac{1}{x};y)\,, \tag{H.808}
$$

so

$$
\mathcal{G}_2 = G(1;x)G(1;y) - G(1;x)G(\frac{1}{x};y)\,, \tag{H.809}
$$

with

$$
\mathcal{S}(\mathcal{G}_2) = (1-x) \otimes (1-y) - (1-x) \otimes (1-xy) + (1-y) \otimes (1-x) - (1-xy) \otimes (1-x)\,. \tag{H.810}
$$

and

$$
T - \mathcal{S}\Big(\mathcal{G}_1 + \mathcal{G}_2\Big) = 0\,. \tag{H.811}
$$

To check the equality on the function level we may choose the point $x = 1$ where

$$
\mathcal{G}_1 + \mathcal{G}_2 = G(0,1;y)\,, \tag{H.812}
$$

and

$$
\lim_{x \to 1} G\left(\frac{(x-1)}{1-y}, \frac{1}{y}; x\right) = G(0, \frac{1}{y}; 1) = G(0,1;y)\,, \tag{H.813}
$$

and we conclude, that we are not missing any term proportional to ζ_2. The important point to keep in mind is, that the ordering on the variables can make a large impact on the size on intermediate expressions. Even more, one will often find that only a small number of integration orders allow for a successful integration.

iv) Perform one of the two remaining two integrations (e.g. in x or y) and outline how you would perform the last integration.

Solution. With the above rewriting e.g. the x-integration becomes trivial,

$$\begin{aligned}
I &= \int_{[0,1]^2} G\left(\frac{(x-1)}{1-y}, \frac{1}{y}; x\right) \frac{\mathrm{d}x}{x+1} \frac{\mathrm{d}y}{y+1} \\
&= \int_{[0,1]^2} \Big[- G\left(\frac{1}{y}, 1; x\right) + G\left(\frac{1}{y}; \frac{1}{y}; x\right) \Big] \frac{\mathrm{d}x}{x+1} \frac{\mathrm{d}y}{y+1} \\
&= \int_{[0,1]} \Big[- G\left(-1, \frac{1}{y}, 1; x\right) + G\left(-1, \frac{1}{y}; \frac{1}{y}; x\right) \Big] \Big|_0^1 \frac{\mathrm{d}y}{y+1} \\
&= \int_{[0,1]} \Big[G\left(-1, \frac{1}{y}, \frac{1}{y}, 1\right) - G\left(-1, \frac{1}{y}, 1, 1\right) - 0 \Big] \frac{\mathrm{d}y}{y+1} \,,
\end{aligned} \tag{H.814}$$

or if we integrate in y first,

$$\begin{aligned}
I &= \int_{[0,1]^2} G\left(\frac{(x-1)}{1-y}, \frac{1}{y}; x\right) \frac{\mathrm{d}x}{x+1} \frac{\mathrm{d}y}{y+1} \\
&= \int_{[0,1]^2} \Big[G(1,x)G(1,y) - G(1,x)G\left(\frac{1}{x}, y\right) + G\left(0, \frac{1}{x}, y\right) \\
&\quad - G\left(1, \frac{1}{x}, y\right) + G\left(\frac{1}{x}, \frac{1}{x}, y\right) \Big] \frac{\mathrm{d}x}{x+1} \frac{\mathrm{d}y}{y+1} \\
&= \int_{[0,1]} \Big[G(1,x)G(-1,1,y) - G(1,x)G\left(-1, \frac{1}{x}, y\right) + G\left(-1, 0, \frac{1}{x}, y\right) - G\left(-1, 1, \frac{1}{x}, y\right) \\
&\quad + G\left(-1, \frac{1}{x}, \frac{1}{x}, y\right) \Big] \Big|_0^1 \frac{\mathrm{d}x}{x+1} \\
&= \int_{[0,1]} \Big[- G\left(-1, \frac{1}{x}, 1\right) G(1,x) + G\left(-1, 0, \frac{1}{x}, 1\right) - G\left(-1, 1, \frac{1}{x}, 1\right) + G\left(-1, \frac{1}{x}, \frac{1}{x}, 1\right) \\
&\quad + \left(\frac{\log^2(2)}{2} - \frac{\pi^2}{12}\right) G(1,x) - 0 \Big] \frac{\mathrm{d}x}{x+1} \,.
\end{aligned} \tag{H.815}$$

To perform the remaining integration one would need to do the same we did before: finding a representation in which the integration variable is only in the argument of the MPLs. This is a bit more work, since the weight is now 3, however the remaining integral is well behaved and will integrate to

$$I = -\frac{1}{2} \log^2(2)\zeta_2 - \frac{3}{8} \log(2)\zeta_3 + \frac{1}{4}\zeta_2^2 \,. \tag{H.816}$$

H.61 Different Representations of Elliptic Curves

Consider the elliptic curve,

$$y^2 = (x - e_i)(x - e_j)(x - e_k) \,, \tag{H.817}$$

with $e_i + e_j + e_k = 0$.

i) Show that by suitable transformation, you can bring it into the form,

$$y_{ij}^2 = x_{ij}\,(x_{ij}-1)\,(x_{ij}-\lambda_{ijk})\;, \tag{H.818}$$

and determine λ_{ijk}, x_{ij} and y_{ij} in terms of x, y, e_i, e_j and e_k. This form is often referred to as Legendre form and written as

$$y^2 = x(x-1)(x-\lambda)\,. \tag{H.819}$$

Solution. We see from the Legendre form, that we want a Möbius transformation to map one root to 0, and one root to 1. So the general transformation which does that is

$$M_{ij} : x \to x_{ij} = (x-e_i)/(e_j-e_i)\,, \tag{H.820}$$

which maps e_i to 0 and e_j to 1. We see that the third root e_k gets mapped to $\lambda_{ijk} = (e_k-e_i)/(e_j-e_i)$. We can also see the action on the terms separately,

$$\begin{aligned} &(x-e_i) = x_{ij}(e_j-e_i), && (x-e_j) = (e_j-e_i)x_{ij}-(e_j-e_i), \\ &(x-e_k) = (e_j-e_i)x_{ij}-(e_k-e_i)\,. \end{aligned} \tag{H.821}$$

So

$$y^2 = (e_j-e_i)^3 x_{ij}(x_{ij}-1)\left(x_{ij}-\frac{(e_k-e_i)}{(e_j-e_i)}\right)\,, \tag{H.822}$$

which tells us $y_{ij} = y(e_j-e_i)^{-3/2}$ gives a Legendre form. Starting from the Weierstrass form, there are six possible Legendre forms which can be obtained by different mappings of the roots. This is just the action of the permutation group S_3 on e_i, e_j and e_k. The associated λ_{ijk} are not independent,

$$\lambda_{jik} = 1-\lambda_{ijk}, \quad \lambda_{ikj} = \frac{1}{\lambda_{ijk}}, \quad \lambda_{jki} = \frac{1}{1-\lambda_{ijk}}, \quad \lambda_{kij} = \frac{\lambda_{ijk}-1}{\lambda_{ijk}}, \quad \lambda_{kji} = \frac{\lambda_{ijk}}{\lambda_{ijk}-1}\,, \tag{H.823}$$

which can be helpful if one wants to remap branch-cut structures.

ii) Show that the transformation $x_{ij} = \xi_{ij}^{-2}$ and $\eta_{ij} = y_{ij}\xi_{ij}^{-3}$ maps the Legendre to the so called Jacobi form,

$$\eta_{ij}^2 = (1-\xi_{ij}^2)(1-\lambda_{ijk}\xi_{ij}^2)\,. \tag{H.824}$$

Solution. Trivial.

H.62 Fundamental Algebraic Relation of the Weierstrass $\wp$-function

i) Prove that the Laurent series for $\wp(z)$ around $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}\,. \tag{H.825}$$

Hint. $\forall\, z$ with $|z|< \omega$, expand $\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2}$ as a power series.

Solution. We have

$$\begin{aligned}\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} &= \frac{1}{\omega^2}\left(\frac{1}{\left(1-\frac{z}{\omega}\right)^2} - 1\right) \\ &= \frac{1}{\omega^2}\sum_{k=1}^{\infty}(k+1)\left(\frac{z}{\omega}\right)^k\,,\end{aligned} \tag{H.826}$$

where we used the derivative of the geometric series,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n\,. \tag{H.827}$$

We substitute it into the definition of the $\wp$-function,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda|\omega\neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \\ &= \frac{1}{z^2} + \sum_{\omega\in\Lambda|\omega\neq 0}\frac{1}{\omega^2}\sum_{k=1}^{\infty}(k+1)\left(\frac{z}{\omega}\right)^k\,,\end{aligned} \tag{H.828}$$

exchange the order of the sums, and use that $\wp(-z) = \wp(z)$. Then only the even powers of ω contribute

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}\sum_{w}(2k+1)\frac{z^{2k}}{\omega^{2k+2}}\,. \tag{H.829}$$

Using the definition of the Eisenstein series,

$$G_{2k} = \sum_{\omega\in\Lambda}\omega^{-2k}\,, \tag{H.830}$$

we get

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}\,. \tag{H.831}$$

ii) $\forall\ z \in \mathbb{C} \setminus \Lambda$, prove that

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6\,. \tag{H.832}$$

Hint. Expand the terms of the relation (H.832) then use holomorphicity and periodicity to show that eq. (H.832) holds $\forall z \in \mathbb{C} \setminus \Lambda$.

Solution. Around $z = 0$,

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty}(2k+1)(2k)G_{2k+2}z^{2k-1}\,, \tag{H.833}$$

i.e.

$$\wp(z) = z^{-2} + 3G_4z^2 + 5G_6z^4 + \dots\,, \tag{H.834}$$

$$\wp'(z) = -2z^{-3} + 6G_4z + 20G_6z^3 + \dots\,, \tag{H.835}$$

$$\wp^3(z) = z^{-6} + 9G_4z^{-2} + 15G_6 + \dots\,, \tag{H.836}$$

$$(\wp')^2(z) = 4z^{-6} - 24G_4z^{-2} - 80G_6 + \dots\,, \tag{H.837}$$

and we see that around $z = 0$,

$$(\wp')^2(z) = 4\wp^3(z) - 60G_4\wp(z) - 140G_6\,. \tag{H.838}$$

Let us now consider the function,

$$f(z) = (\wp')^2(z) - \left(4\wp^3(z) - 60G_4\wp(z) - 140G_6\right)\,. \tag{H.839}$$

It is holomorphic at $z = 0$, with $f(0) = 0$. But $\wp(z)$ is an elliptic function (double periodic $\wp(z) = \wp(z+w)$ $\forall w \in \Lambda$ and meromorphic in $\mathbb{C}$) and absolutely and uniformly convergent on $\mathbb{C} \setminus \Lambda$, so $f(z)$ is a double periodic function with the fundamental parallelogram Λ, which is holomorphic at $z = 0$ and on $\mathbb{C} \setminus \Lambda$. Thus, $f(z)$ is a bounded entire function, which, by Liouville's theorem, can only be a constant. The value of the constant is e.g. determined at $z = 0$, as we did above: from $f(0) = 0$ follows $f(z) = 0$ for all $z \in \mathbb{C}$. Which proves

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6\,. \tag{H.840}$$

H.63 Period Integrals (Optional)

Consider the elliptic curve in Legendre-form and the periods shown in fig. H.23.

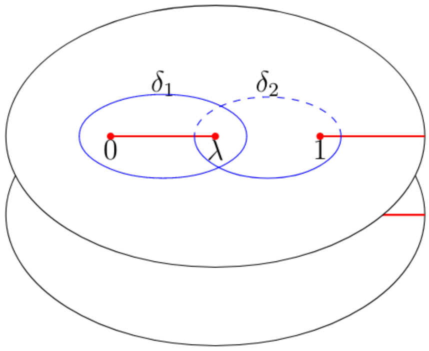


Figure H.23: The period integrals of an elliptic curve. Dashed lines show parts of the contour which are on the second Riemann sheet.

We now want to study the period,

$$\Psi_1 = \int_{\delta_1} \eta \,, \tag{H.841}$$

where $\eta = \mathrm{d}y = \dfrac{\mathrm{d}x}{\sqrt{x}\sqrt{\lambda - x}\sqrt{1-x}}$ and assume $0 < \lambda < 1$. Show that this period integral can be written in terms of the complete elliptic integral of the first kind defined in the lecture.

Solution. The contour δ_1 encircles the branch-cut $(0, \lambda)$ in counter-clockwise direction. If we now consider gluing the boundary of $\mathbb{C}$, we will get the sphere $\mathbb{CP}^1$ with 2 slits, discussed in the lecture, which is easier for visualizing the transformations we want to perform. Consider now dragging the contour δ_1 behind the sphere, such that it encircles the branch-cut $(1, \infty)$ clockwise. You can of course deform the contour, as long as you don't cross any branch-cuts.

So the period integral may be written as

$$\Psi_1 = \int_{\delta_1} \eta = \int\limits_{\infty - i\epsilon}^{1 - i\epsilon} \mathrm{d}y + \int\limits_{1 + i\epsilon}^{\infty + i\epsilon} \mathrm{d}y = - \int\limits_{1 - i\epsilon}^{\infty - i\epsilon} \mathrm{d}y + \int\limits_{1 + i\epsilon}^{\infty + i\epsilon} \mathrm{d}y \,, \tag{H.842}$$

where we opened the contour at the endpoints. The first integral is ϵ below the $(1, \infty)$ branch cut and the second one is slightly above. However, the square root has opposite sign above and below the branch-cut, so we can summarize our integral e.g. as

$$\Psi_1 = -2 \int\limits_{1 - i\epsilon}^{\infty - i\epsilon} \mathrm{d}y \,. \tag{H.843}$$

There is a second point we need to take into account. The argument of the roots,

$$\frac{\mathrm{d}x}{\sqrt{x}\sqrt{\lambda - x}\sqrt{1-x}} \,, \tag{H.844}$$

is negative for $x > 1$, where we want to integrate. If we want positive arguments for $x > 1$, we have to write

$$\frac{\mathrm{d}x}{\sqrt{x}\sqrt{\lambda - x}\sqrt{1-x}} = \frac{\mathrm{d}x}{\sqrt{x}i\sqrt{x-\lambda}i\sqrt{x-1}} = -\frac{\mathrm{d}x}{\sqrt{x}\sqrt{x-\lambda}\,,\sqrt{x-1}} \tag{H.845}$$

which basically determines on how we decide to evaluate on the real line ($y(x)$ is only defined on $\mathbb{C}$ minus the branch cuts). So we have

$$\Psi_1 = -2 \int\limits_{1-i\epsilon}^{\infty - i\epsilon} \mathrm{d}y = 2\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}\sqrt{x-\lambda}\sqrt{x-1}} = 2\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x(x-\lambda)(x-1)}}\,. \tag{H.846}$$

The last thing we need is the transformation to Jacobi form, $x = t^{-2}$,

$$\Psi_1 = -4\int_1^0 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\lambda t^2)}} = 4\int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\lambda t^2)}}\,, \tag{H.847}$$

which gives the complete elliptic integral of the first kind defined in the lecture.

In general, the detailed analysis we performed here will often not be necessary for pure mathematical considerations. There the only thing which is important is, that you have a single valued determination on $\mathbb{C}$ minus the branch-cuts. How you implement it, is only of practical concern. However in physics we have a defined way on how we approach branch-cuts: it is given by sending Feynman's $i\epsilon$ to 0. If we want to get the correct imaginary parts for Feynman integrals, we have to be careful with how we perform analytic continuations and how we treat square roots. In particular, if you have square roots with negative arguments, you will find that the imaginary part assigned by e.g. Mathematica[9] will not be the imaginary part dictated by Feynman prescription.

[9] It is documented how they compute on branch-cuts and which branch-cuts they take for every function to obtain a single valued version.

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