Channeling 2010, Ferrara, October 3-9

Unitarity and causality bound on the energy radiated by a charged particle passing by a periodic radiator

Xavier Artru

Institut de Physique Nucléaire de Lyon,

Université de Lyon, CNRS- IN2P3

and Université Lyon-1, France.

Outline

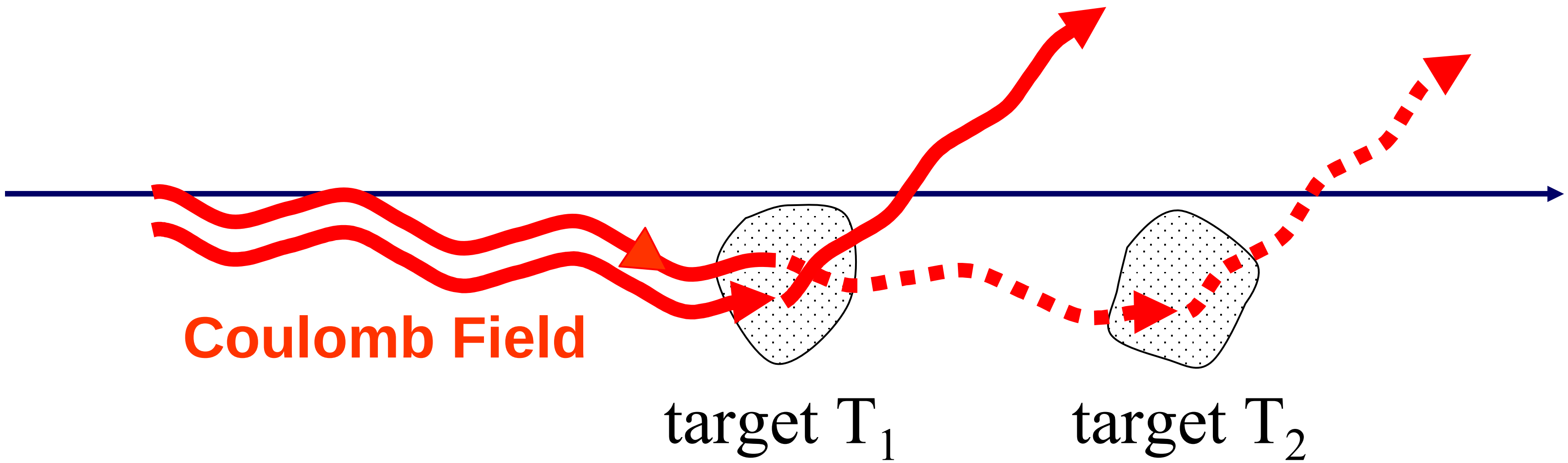
1) ***THE SHADOWING ARGUMENT***

- Shadowing of the Coulomb field.
- How it reduces the Smith-Purcell radiation
- Rough estimation of the bound

2) ***PRECISE CALCULATION OF THE BOUND***

- Retro-action of the radiator due to the reflection of the Coulomb field
- Exponential bound on the reflection coefficient for the ***evanescent*** wave
- Bounds for differential spectrum and the total energy loss.
- Appendix : S-matrix derivation of $|R|\leq 1$ for evanescent waves

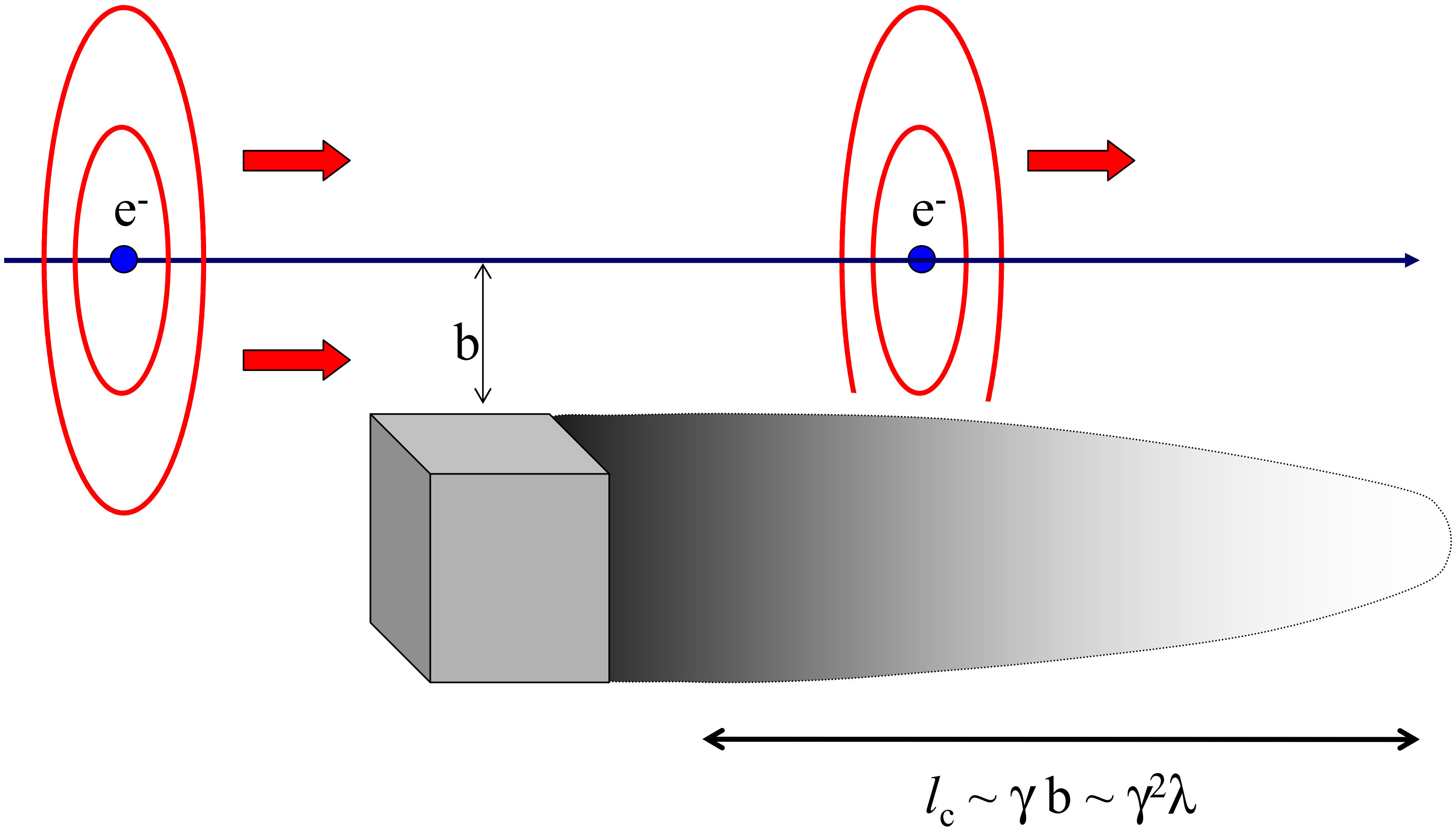
Shadowing in Diffraction Radiation (DR)



- DR = *scattering of quasi-real photons from the particle field.*

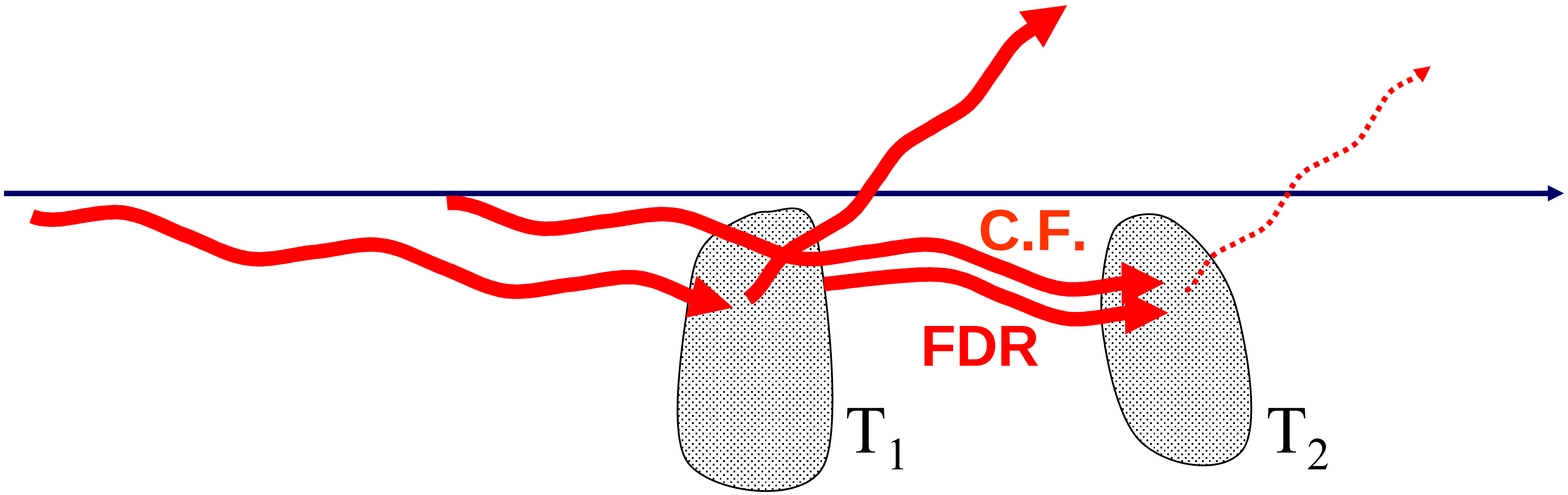
- Here we have **two** targets.
- T_2 is in the **shadow** of T_1 ,
- therefore radiates **less** than if it was alone.

Extension of the shadowing region



After the distance l_c the Coulomb field is fully restored.

Another point of view



The *Coulomb field* of the particle (**C.F.**) and the *Forward Diffracted Radiation* (**FDR**) from T_1 have opposite phases $\rightarrow$ they ***interfere destructively*** in T_2

→ Shadow can be understood as a *rescattering* effect

~~Experimental proof of shadowing:~~

RREPS-2009, Moskow, Sept 7-11

“Shadowing” of the electromagnetic field of relativistic charged particles

X.Artru[1], G. Naumenko[2], A. Potylitsyn[3], Yu. Popov[3], and L. Sukhikh[3]

[1] *Institut de Physique Nucléaire de Lyon, Université de Lyon,CNRS- IN2P3 and Université Lyon 1, 69622 Villeurbanne, France.*

[2] *Nuclear Physics Institute of Tomsk Polytechnic University, Russia.*

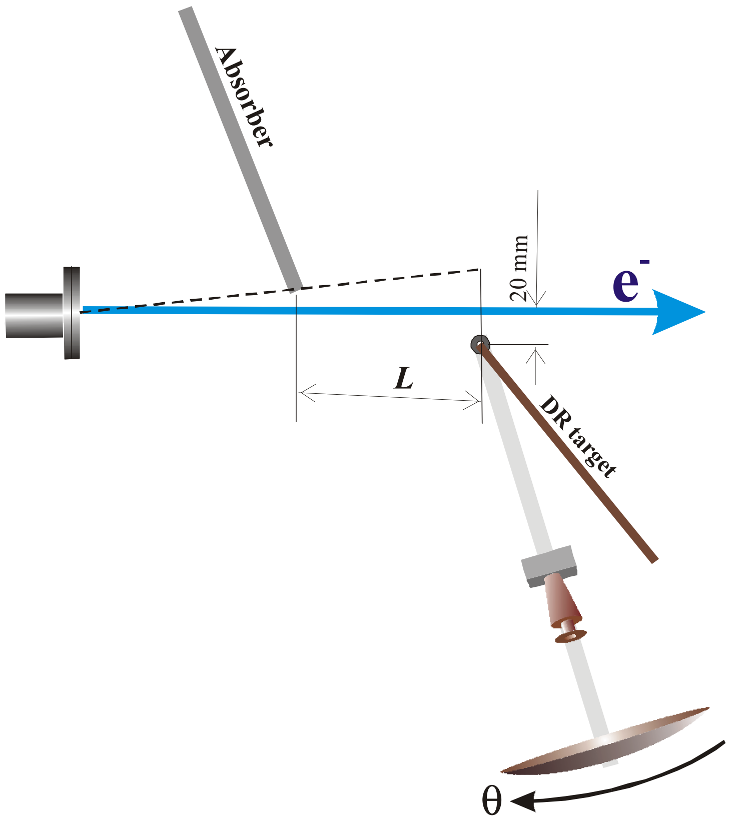
[3] *Tomsk Polytechnic University, Russia.*

Two screen configurations

a) **opposite sides :**

normal DR

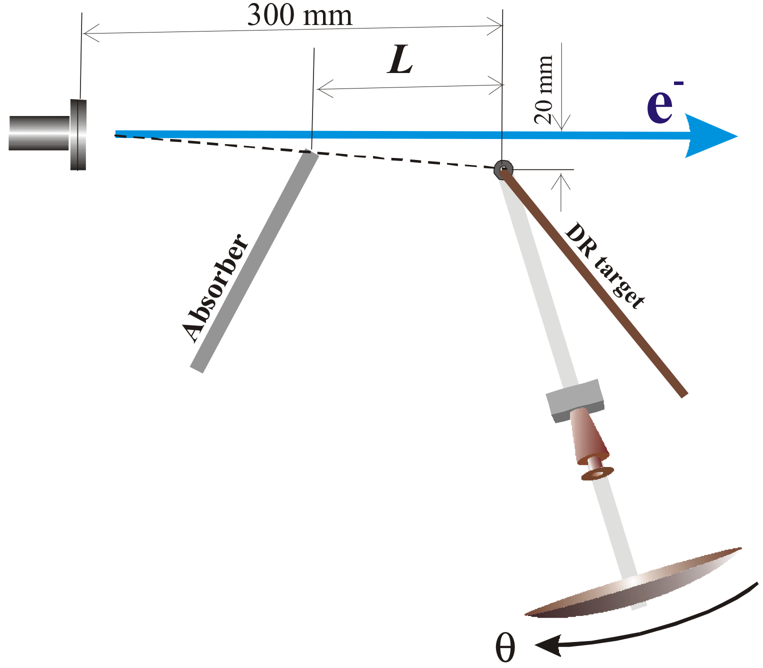
→ no shadowing



b) **same side :**

reduced DR

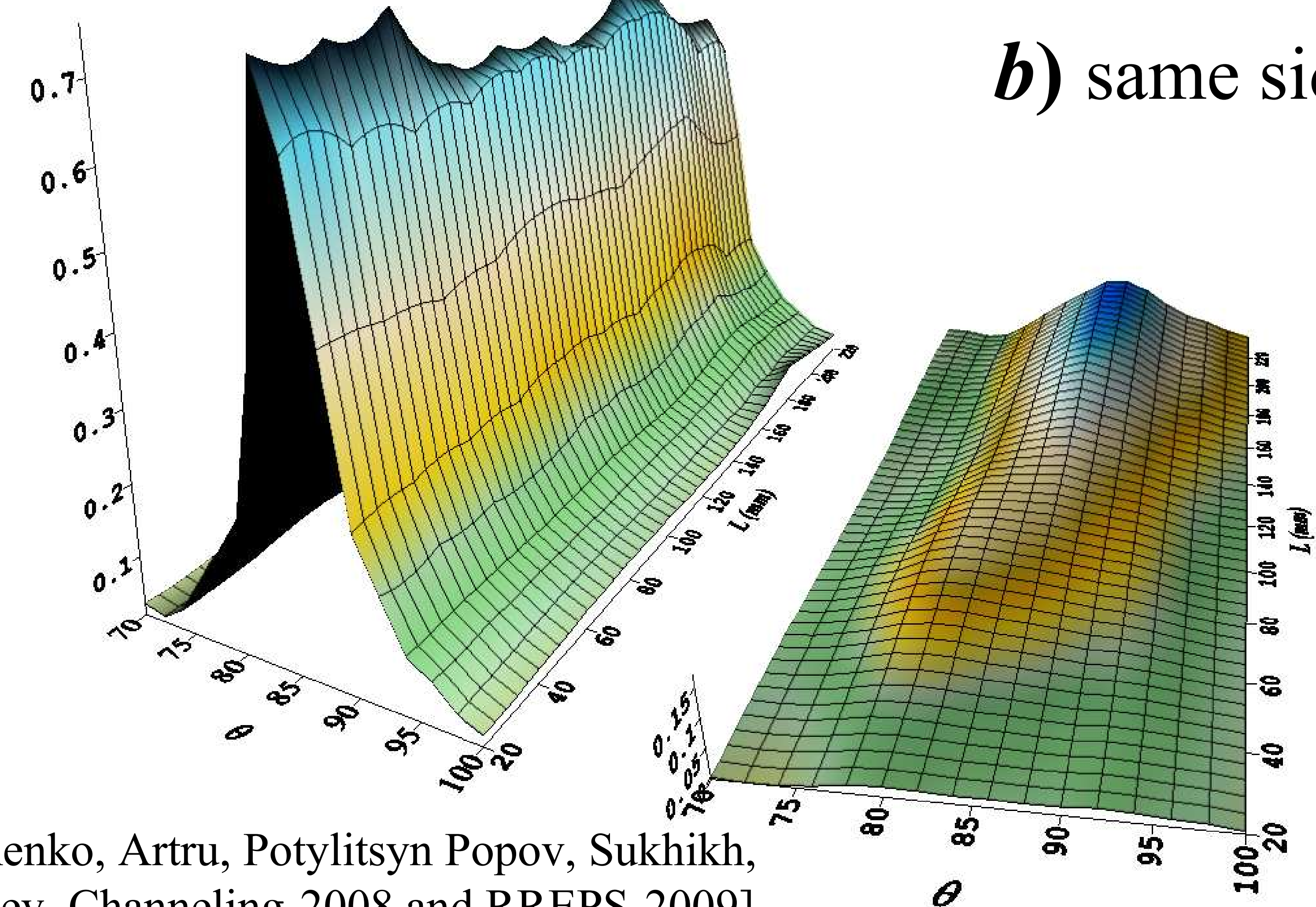
→ shadowing



measured angular distribution

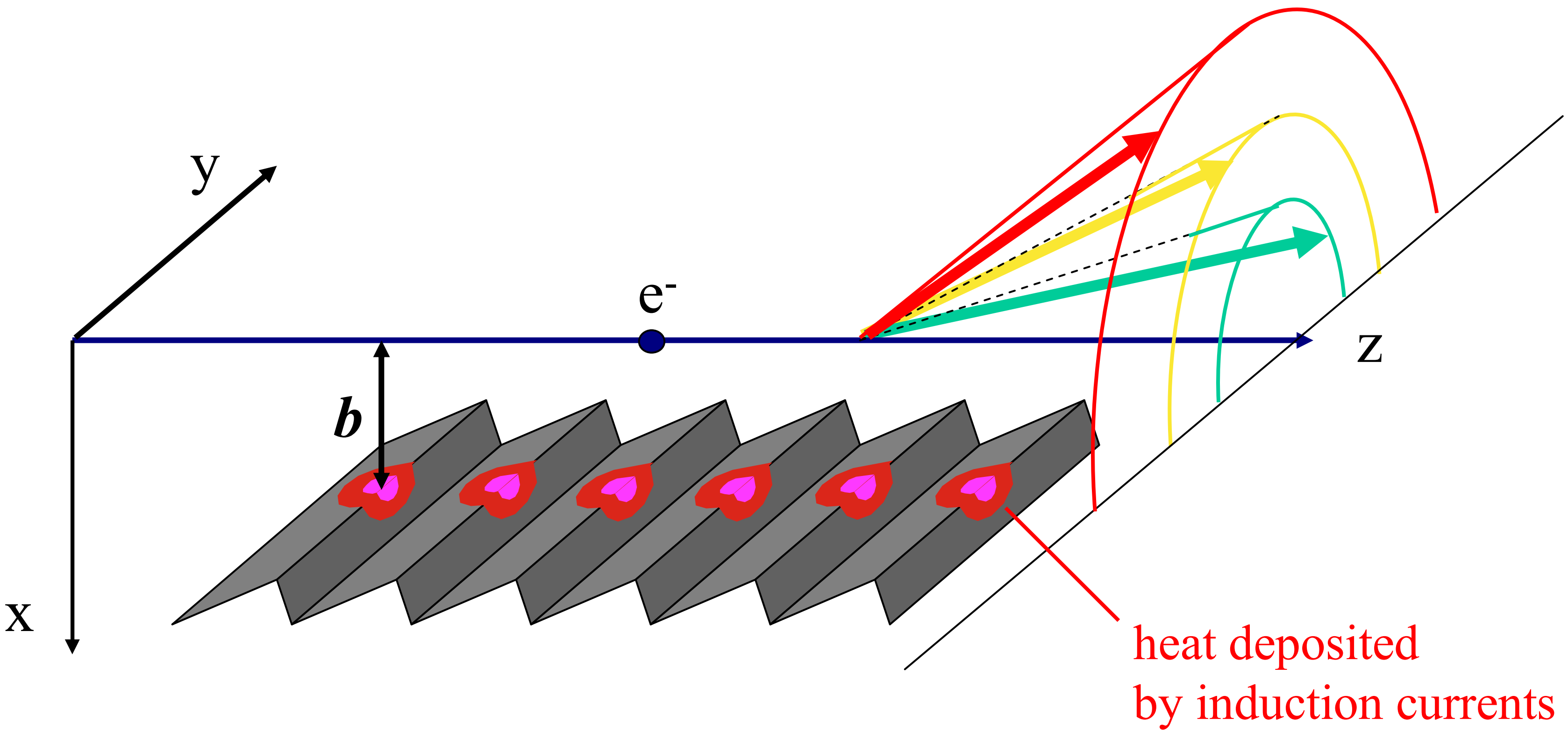
***a*) opposite sides**

***b*) same side**



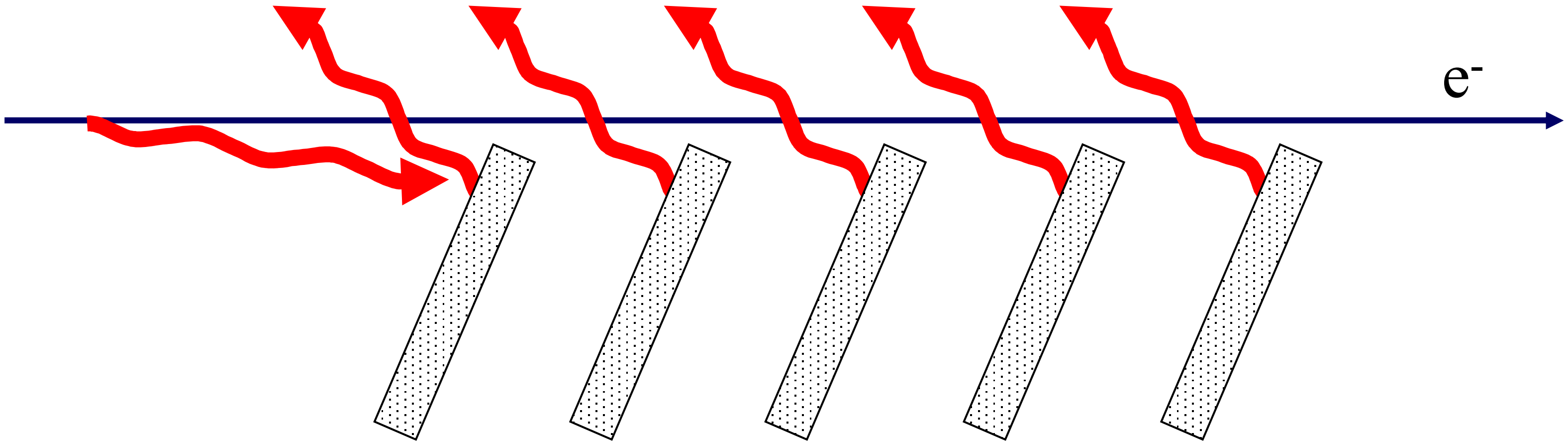
[Naumenko, Artru, Potylitsyn Popov, Sukhikh, Shevelev, Channeling-2008 and RREPS-2009]

Smith-Purcell radiation : geometry



- ***b*** *= impact parameter*
- the target is periodic in z, uniform in y
- x- axis is toward the target

Shadow effect in Smith-Purcell radiation



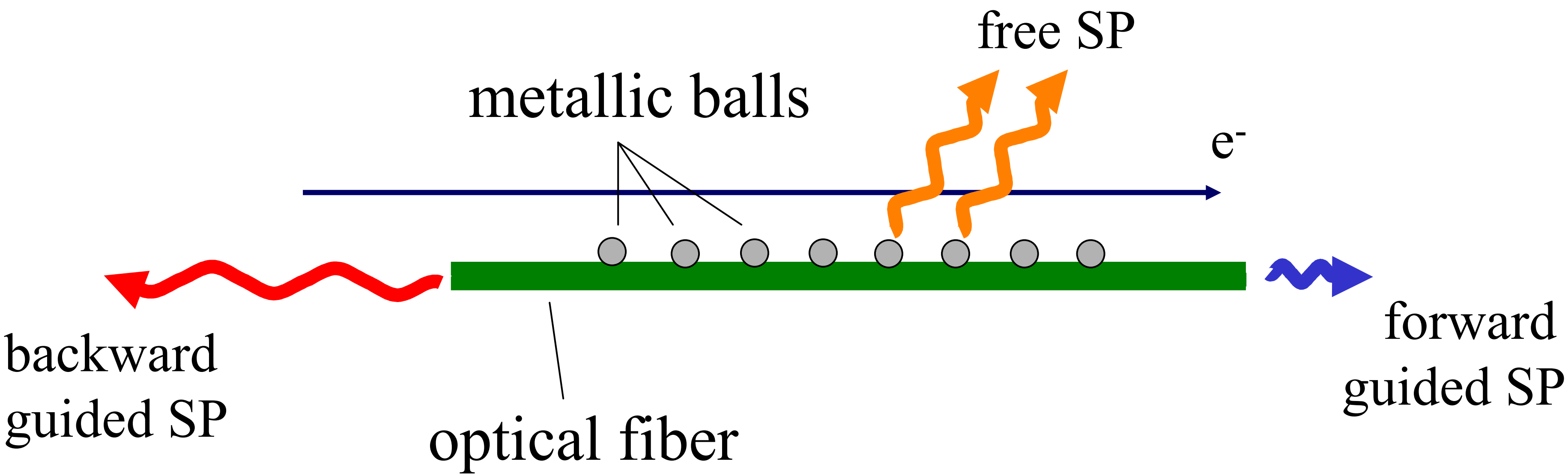
example of SP radiator : a periodic set of foils.

If we take the **addition** of the **single-foil DR amplitudes**, we ignore the shadow effect, therefore we **over-estimate** the SP yield.

→ Shadowing suggests an **upper bound** on Smith-Purcell radiation

Another example of Smith-Purcell radiator

[X.A. & C. Ray, RREPS-2007 - arXiv:0803.0249]



- a *free* or *guided* SP beam can be enhanced by a *plasmon resonance* of the balls.
- intensities are reduced by ***shadowing*** between the balls.

First rough estimation of the bound for SP radiation

- Total energy in *Diffraction Radiation* from a **single foil**:

$$W_1 = 3/8\,\gamma\,\alpha/b \qquad (\text{units } \hbar\text{barre}=c=1\ ;\ \alpha=1/137\,)$$

- optimum spacing = minimum distance to « repair » the Coulomb field:

$$L_{min} = L_c \sim \gamma\, b \qquad (\textit{order of magnitude})$$

$\rightarrow$ bound $dW/dz \leq W_1 / L_{min} \sim 3/8\,\alpha\,/b^2$ (*order of magnitude*)

- This bound is independent on γ, on the radiator material
- it applies to the *total* energy loss,

W = **radiated** energy + **absorbed** energy.

Preliminary conclusion

There should be a universal bound on the *braking force,*

$$dW/dz \leq C\,\alpha/b^2$$, where C is of the order of unity,

independent of - the spatial period and material of the radiator,
- the electron momentum

The bound applies to the *total* energy loss,

W = **radiated** energy + **absorbed** energy.

For an electron passing though a *cylindrical corridor* of radius *r*, we expect

$$dW/dz \leq C'\,\alpha/r^2 , \text{ with } C' > C$$

R. Chehab : such a bound may apply to electrons passing trough iris diaphragms in accelerating cavities.

Exact calculations.
Coulomb and reflected fields

Particle Coulomb field :

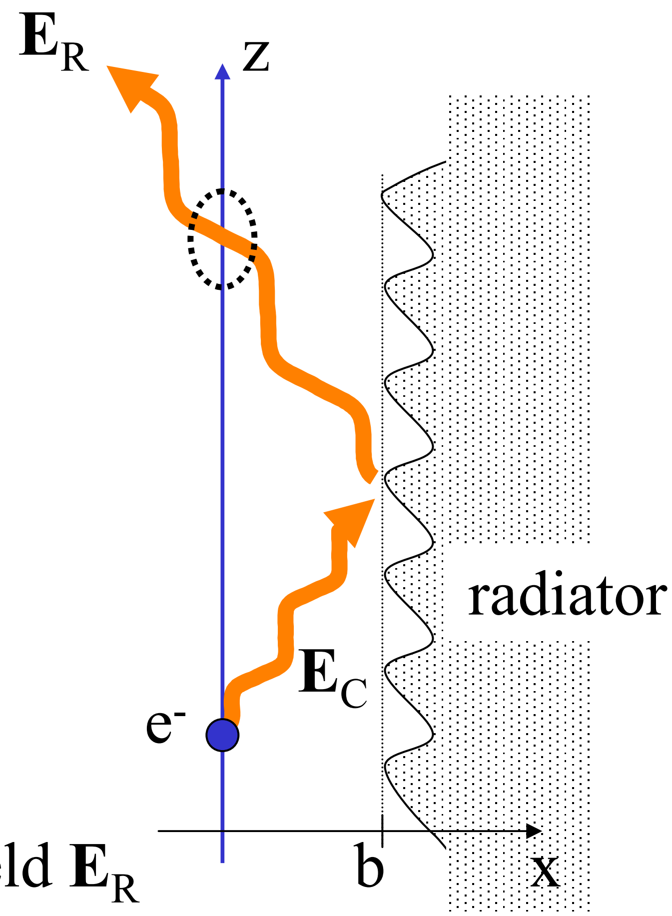
$$\mathbf{E}_C(t,x,y,z) = (e/4\pi)\,\gamma$$

$$\times\ [(x^2+ y^2+ \gamma^2(z\text{-}vt)^2]^{-3/2}\ \{ x , y , z\text{-}vt \}$$

Total field in the region $x < b$:

$$\mathbf{E} = \mathbf{E}_C + \mathbf{E}_R$$

The particle is decelerated by the *reflected* field $\mathbf{E}_R$



Space-time periodicity

$$\mathbf{E}(t,x,y,z) = \mathbf{E}(t+L/v,\ x,\ y,\ z+L) \qquad (\mathbf{E} = \mathbf{E}_C \text{ or } \mathbf{E}_R)$$

$\downarrow$ e^- travelling time between two groves

$\downarrow$ spatial radiator period

In 4-momentum space (ω, k_x, k_y, k_z), we have discrete k_z :

$$k_z = \omega/v + n\,G\,, \qquad \text{with } G = 2\pi/L$$

For the Coulomb field : $n = 0$, $k_z = \omega/v$

$\Downarrow$

(phase velocity ω/k_z = electron velocity)

Klein-Gordon equation in momentum space

$$\omega^2 = k_x{}^2 + k_y{}^2 + k_z{}^2$$

$$\rightarrow \quad k_x = \pm\,(\omega^2 - k_y{}^2 - \underbrace{k_z{}^2}_{\omega/v + nG})^{1/2}$$

The *sign* of $k_x{}^2$ depends on n, ω and k_y

$k_x{}^2 > 0$: k_x = real ; for Smith-Purcell radiation, $k_x > 0$

$k_x{}^2 < 0$: k_x = pure imaginary $= \pm\, i\,\kappa$ (evanescent wave)

$\Downarrow$

This is the case for $n = 0$, with $\kappa = (k_y{}^2 + \omega^2/(\gamma v)^2\,)^{1/2}$

Fourier decomposition in the region $x \in [0,b]$

$$\mathbf{E}^{(\mathbf{TM})}(t,x,y,z) = (2\pi)^{-2} \int_{-\infty}^{+\infty} d\omega \;\exp(-i\omega t)$$

$$\sum_{n=-\infty}^{+\infty} \exp(ik_z z) \quad \int_{-\infty}^{+\infty} dk_y \;\exp(ik_y y)$$

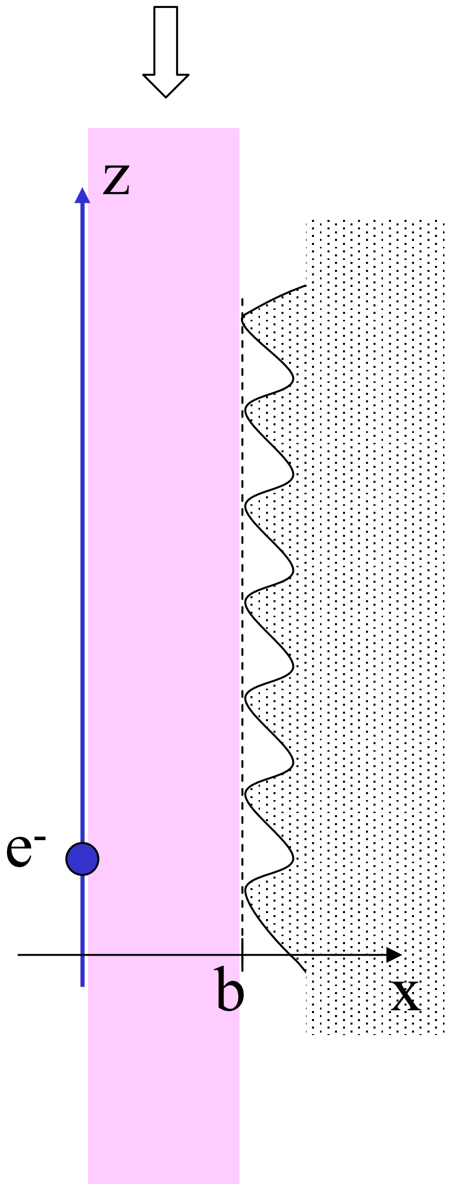
$$\sum_{\text{sign of kx}} \exp(ik_x x) \quad \underbrace{\{ -k_x k_z\,,\; -k_y k_z\,,\; \mathbf{k}_T^2 \}}_{\text{TM polarisation vector}}$$

$$\times F(\omega,\; k_x, k_y, n)$$

with $k_z = \omega/v + nG$

$k_x = \pm(\omega^2 - k_y^2 - k_z^2)^{1/2}$,

$\mathbf{k}_T^2 = k_x^2 + k_y^2$



TM = ‘‘transverse magnetic’’

(there is a similar formula for the TE polarisation)

Various reflected waves (denoted "R-waves")

Evanescent R-wave

$k_x = - i\,\kappa$ (evanescent toward left).

The n=0, R-wave decelerates the electron.

Escaping R-wave (Smith-Purcell)

k_x real,

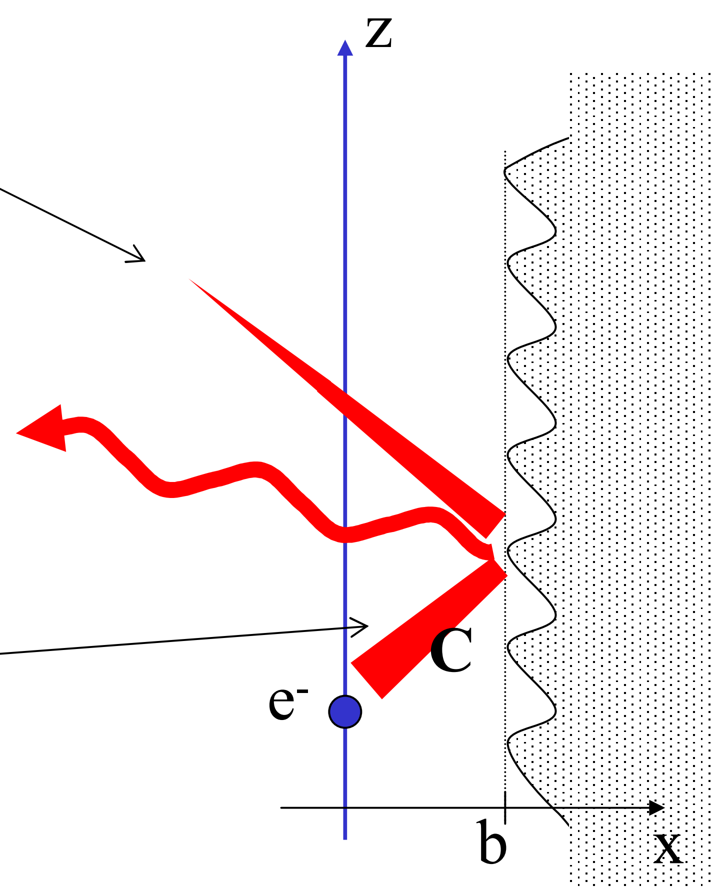
$k_x/\omega < 0$ (propagating to the left)

Coulomb field

$n = 0 \rightarrow k_z = \omega/v$;

$k_x = + i\,\kappa$ (evanescent toward right),

with $\kappa = (k_y^{\,2} + \omega^2/(\gamma v)^2\,)^{1/2}$



The reflection matrix

The reflection can change k_z (*diffraction*) : the reflection coefficient is a *matrix* $\langle n' | \mathrm{R} | n \rangle$ in the discrete k_z space.
The polarisation, ε = TM or TE, can also change. The full matrix writes

$$\langle n', \varepsilon' | \mathrm{R}\,(\omega, k_y) | n, \varepsilon \rangle \qquad \text{(multi-channel reflection)}$$

The Coulomb field and the decelerating wave have both $n = 0$, ε = TM
$\rightarrow$ we are interested only in the *diagonal element*

$$\langle 0, \mathrm{TM} | \mathrm{R}\,(\omega, k_y) | 0, \mathrm{TM} \rangle$$

The decelerating wave is then

$$F_R(\omega,\ k'_x = -i\kappa\,,\ k_y,\ n=0) \ = \ R\,(\omega, k_y) \ \ F_C(\omega,\ k_x = i\kappa\,,\ k_y,\ n=0)$$

with $\kappa = (k_y^{\,2} + \omega^2/(\gamma v)^2\,)^{1/2}$

Preliminary expression for the deceleration

- Energy loss $\langle dW/dz \rangle = e \langle E_z \rangle = e\,(2\pi)^{-2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dk_y$
  $\times\ \mathbf{k}_T^2\ R(\omega, k_y)\ \ F_C(\omega,\ k_x=i\kappa\ ,\ k_y\ , n=0)$

- $n=0 \ \rightarrow\ \mathbf{k}_T^2 = -\ \omega^2/(\gamma v)^2$ and $\kappa = (k_y^2 + \omega^2/(\gamma v)^2\ )^{1/2}$

- Fourier amplitude of the Coulomb field :

$$F_C(\omega,\ k_x=i\kappa\ ,\ k_y\ ,\ n=0)\ =\ -\ ie/(2\omega\kappa)$$

$$\rightarrow\quad \langle dW/dz \rangle = 2ie^2\,(4\pi\,\gamma v)^{-2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\omega\, dk_y\ \ (\omega/\kappa)\ \ \boldsymbol{R}(\omega, \mathbf{k_y})$$

We just need a bound on this reflection coefficient….

Reduction to positive frequencies

E(t,x,y,z) is *real*. For the Fourier components, it implies

$$f(\omega,\ k_x\,,\ k_y,\ n) = f^{\,*}(-\omega,\ -k^*_x\,,\ -k_y\,,\ -n)$$

It allows us to use only **positive** frequencies :

$$\langle dW/dz\rangle = -\ e^2\,(2\pi\,\gamma v)^{-2} \int_{\mathbf{0}}^{+\infty} d\omega \int_{-\infty}^{+\infty} dk_y\quad (\omega/\kappa)\ \ \mathrm{Im}\, R(\omega,k_y)$$

Exponential damping of the reflection coefficient

$R(\omega, k_y)$ is defined as (reflected wave) / (incoming wave), **where the waves are measured at $\boldsymbol{x=0}$.**
If we measure the waves at $\boldsymbol{x=b}$, we have another coefficient,

$$R_{\mathbf{b}}(\omega, k_y) = \exp(-2ik_x b)\, R(\omega, k_y)$$

For $k_x = i\kappa$, $$R_{\mathbf{b}}(\omega, k_y) = \exp(2\kappa b)\, R(\omega, k_y)$$

For *real* k_x we have the **unitarity bound**

$$|R(\omega, k_y)| \leq 1, \quad |R_{\mathbf{b}}(\omega, k_y)| \leq 1$$

Causality $\rightarrow$ *analyticity* in ω $\rightarrow$ this bound extends to *evanescent* waves (details in appendix later). Therefore

$$\boldsymbol{|R(\omega, k_y)| \leq \exp(-2\kappa b)}$$

Final results for the bound*s*

$$\langle dW/dz \rangle \leq (\alpha/\pi)\,(\gamma v)^{-2} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} dk_y \,(\omega/\kappa)\, \exp(-2\kappa b)$$

$(\alpha = e^2/4\pi = 1/137)$

→ *frequency-angle* spectrum of Smith-Purcell radiation

$$\langle\, d^3W/dz\, d\omega\, dk_y \,\rangle \leq (\alpha/\pi)\,(\gamma v)^{-2}\,(\omega/\kappa)\, \exp(-2\kappa b)$$

→ *frequency* spectrum

$$\langle\, d^2W/dz\, d\omega \,\rangle \leq (2\alpha/\pi)\,(\gamma v)^{-2}\, \omega\, K_0(2\omega b/\gamma v)$$

→ *total energy loss* (radiation + heating by induction currents)

$$\langle\, dW/dz \,\rangle \leq \alpha/(2\pi)\, b^{-2}$$

The bound in practical units

The maximum braking force $F_{max} = \alpha/(2\pi)\, b^{-2}$ is the same as produced by a fictitious positron at distance $(2\pi)^{1/2}\, b$ behind the electron, in the electron rest frame.

For $b = a_0$ (Bohr radius), this braking power is 0.82 GeV/cm.
For b = 1 mm, it becomes 0.23 eV/kilometre

An electron beam of 1 ampere, flying 1 mm above a radiator, delivers a maximum (radiated or absorbed) linear power of $2.3\ 10^{-4}$ watt per metre (if there is no microbunching).

Appendix : generalisation of $\left|R_b(\omega, k_y)\right| \leq 1$ to evanescent waves

1) Check with the Schrödinger equation for a step-wise potential

propagating incoming wave :

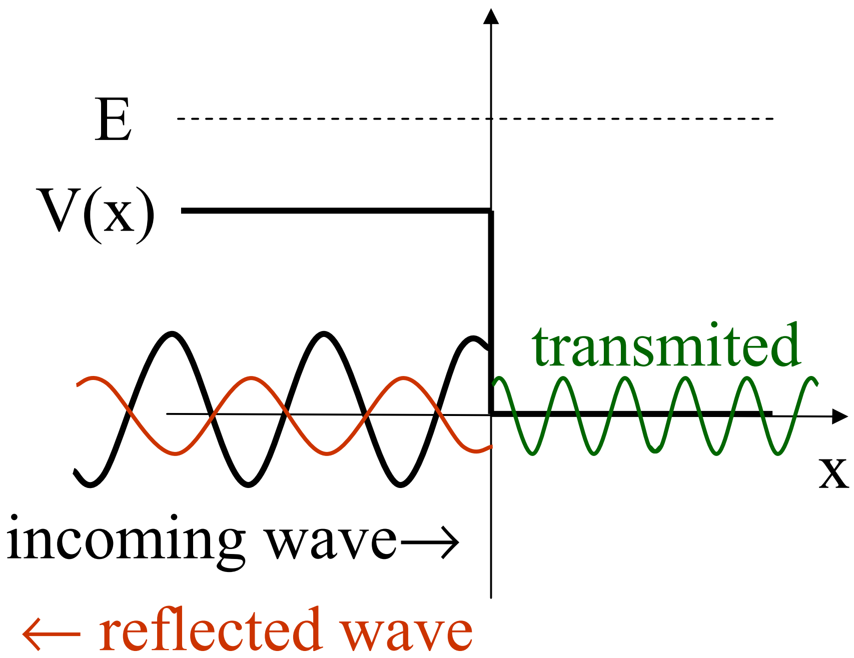
$R(E) = (k_{in} - k_{tr}) / (k_{in} + k_{tr})$

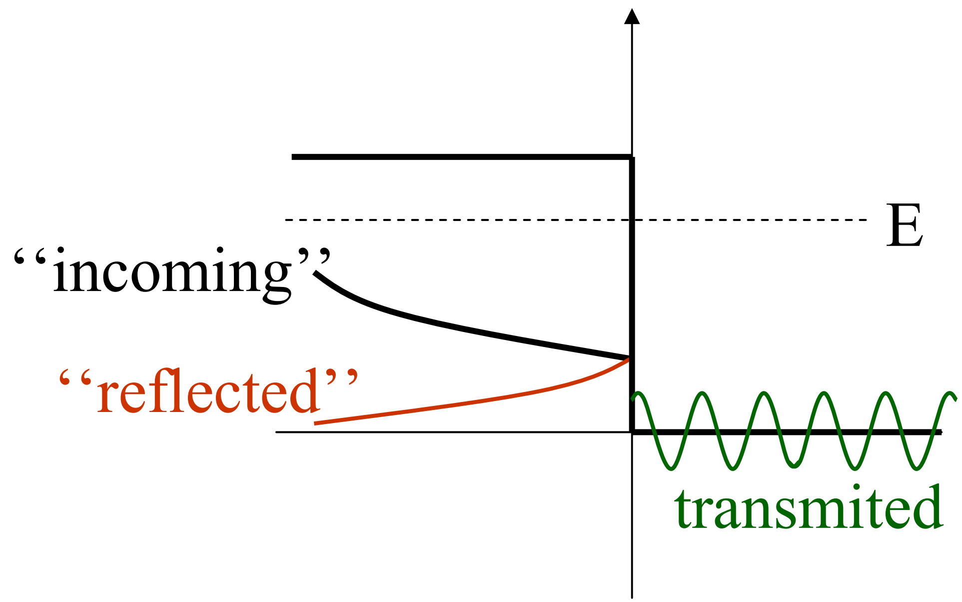
$\rightarrow \left|R(E)\right| \leq 1$

evanescent "incoming" wave :

$R(E) = (\kappa + ik_{tr}) / (\kappa - ik_{tr})$

$\rightarrow \left|R(E)\right| = 1$





2) "S-matrix" proof

- fix k_y and k_z ; re-write k_x as $\mathbf{k} = i\,(M^2 - \omega^2)^{1/2}$, with $M^2 = k_y^{\,2} + k_z^{\,2}$
- call **R(ω)** the reflection coefficient for $(k_x , k_y , k_z) \rightarrow (-k_x , k_y , k_z)$

Causality : a reflected signal cannot exist **before** the incident signal

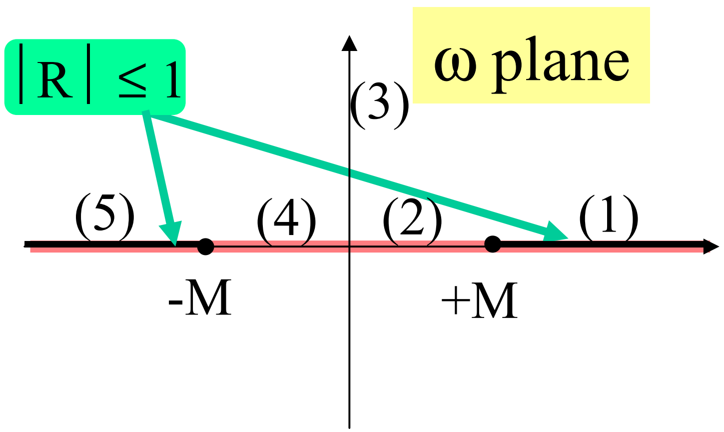
→ R(ω) is **analytic** in the half-plane Im(ω) > 0

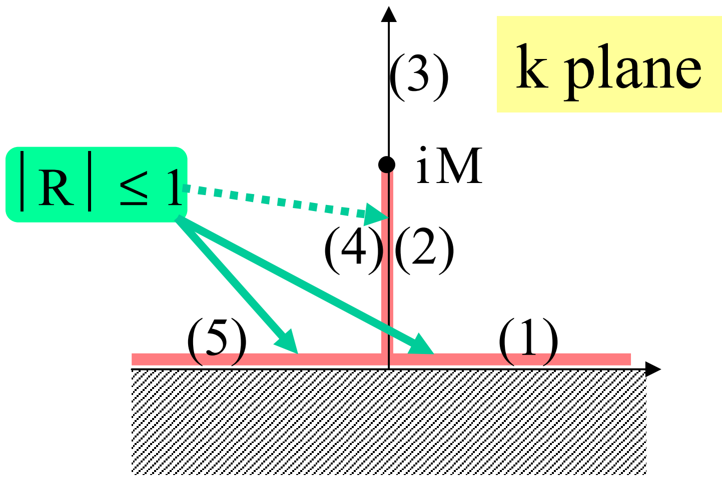
→ analytic in **k** in the **half-plane Im(k) > 0**

.... EXCEPT PERHAPS ON THE VERTICAL PINK SEGMENT !...

- R is a **continuous** in k (to check) → R analytic in k in the **whole** half-plane
- $R \rightarrow 0$ when $\omega \rightarrow \infty$
- $|R(\omega)|$ is maximum on the **frontier** of the analytical domain

then $|R(\omega)| \leq 1$ **in the whole half-plane Im(k) > 0**





About the role of causality

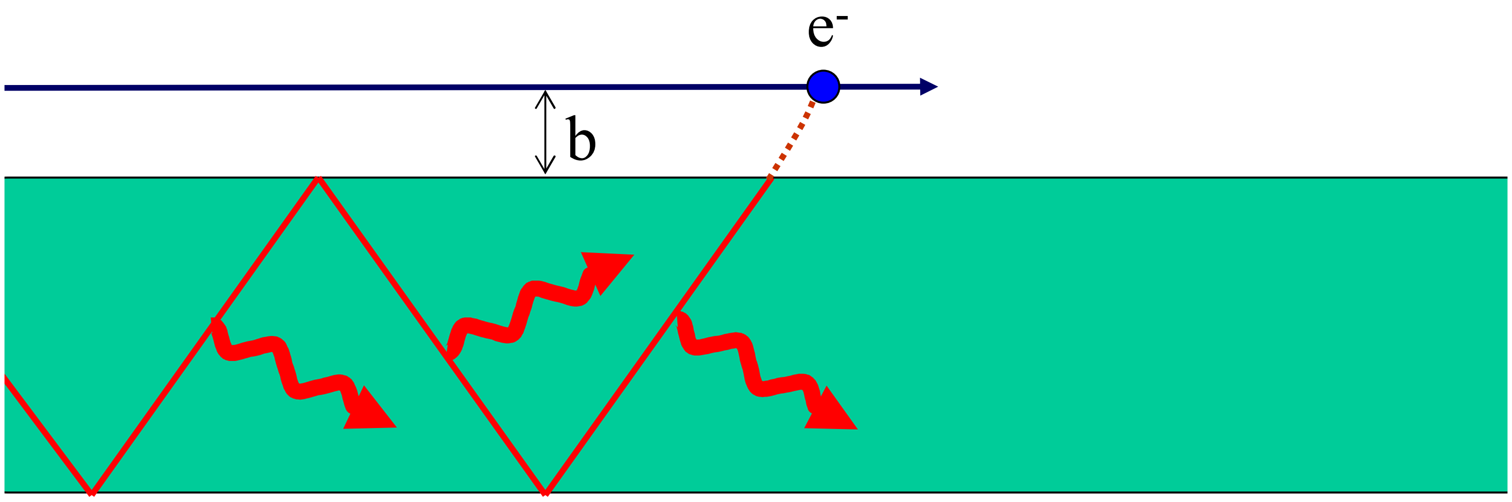
Relativistic causality : no signal, no influence can be faster than c.
$\rightarrow$ The reaction of the radiator medium occurs when the electron has almost passed. This is one of the deep reasons for the bound.

Military comparison : *To stop the invasion by a foreign aircraft, our defense needs fast communications and weapons. Unfortunately, our communications and bullets have finite speed c. This fact limits the stopping power of our defense*

foreign aircraft $\leftrightarrow$ incoming electron
soldiers $\leftrightarrow$ electrons of the target material
communications, bullets $\leftrightarrow$ photons
stopping power $\leftrightarrow$ energy loss

Questions and perspectives

- Is the bound $\langle\, dW/dz \,\rangle \;\leq\; \alpha/(2\pi)\, b^{-2}$ the ultimate one ?
- Can it be saturated ? (e.g., by an electron flying over a resistive plane ?)
- if the radiator is *non-uniform in y* : same bound or a weaker one ?
- Find the bound for a *cylindrical corridor* $\rightarrow$ application to accelerating structures (Chehab's suggestion) and to hyper-channeled particles (e.g. the uranium ions of C. Ray's talk).
- Check the bound for the « distant Cherenkov » effect



Résumé

1. **The shadowing of the electron electromagnetic field by a piece of matter, experimentally established, suggests a bound in αb^{-2} for Smith-Purcell radiation power.**

2. **From unitarity and causality, exact expressions for this bound have been obtained,**
   - for the differential spectrum in ω or (ω, k_y)
   - for the total energy loss or ‘‘braking force’’ dW/dz

1. **The braking force bound $\alpha/(2\pi b^2)$ includes the energy deposit in the target and, for a dielectric target, the distant Cherenkov effect.**

Thank you for your attention and patience !